3450:438/538:001 **Homework 14** Fall 2007

Course: Advanced Engineering Math I
Instructor: Dr. Laura Gross
Optional due date: Friday, November 30, 2007

IF YOU CHOOSE TO SUBMIT THIS HOMEWORK, THE GRADE WILL REPLACE YOUR LOWEST HOMEWORK GRADE (provided it is higher than your lowest homework grade to date).

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1. Prove that $\text{curl}\,(\nabla f) = \mathbf{0}$ if $f(x, y, z)$ has continuous second-order partial derivatives.

* Note this implies that if $\text{curl}\,\mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is not conservative!

* Fact: If $\text{curl}\,\mathbf{F} = \mathbf{0}$ and the component functions of $\mathbf{F}$ have continuous second partial derivatives, then $\mathbf{F}$ is conservative.

2. Consider the vector field $\mathbf{F}(x, y, z) = e^y\mathbf{i} + xe^y\mathbf{j} + (z + 1)e^z\mathbf{k}$ and the curve $\mathcal{C}$: $x = t$, $y = t^2$, $z = t^3$, $0 \leq t \leq 1$.
   (a) Find a function ϕ such that $\mathbf{F} = \nabla\phi$.
   (b) Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ along $\mathcal{C}$.

3. Determine whether or not the vector field

$$\mathbf{F}(x, y) = 3z^2\mathbf{i} + \cos(y)\mathbf{j} + 2xz\mathbf{k}$$

   is conservative. If it is, find a function ϕ such that $\mathbf{F} = \nabla\phi$.

4. Use the definition of the gradient and the divergence to write an expression for $\nabla \cdot \nabla\phi$, where $\phi(x, y, z)$ is a scalar function. (Alternate notations for the expression—called the Laplacian of ϕ—is $\nabla^2\phi$ or $\Delta\phi$.)

5. Prove that $\text{div}\,(\text{curl}\,\mathbf{F}) = 0$ if $\mathbf{F}$ is a vector field on R^3 whose component functions have continuous second-order partial derivatives. (If $\mathbf{F}$ is the velocity field of a fluid, then $\text{div}\,(\text{curl}\,\mathbf{F}) = 0$ means there is no flow along the axis of rotation of the fluid.)

6. Let $\mathbf{F} = yz\mathbf{i} + xyz\mathbf{j} + -x^2\mathbf{k}$. Show whether there is a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$ on all of $\mathcal{R}^3$.