3450:438/538:001 **Homework 12** Fall 2007

Course: Advanced Engineering Math I
Instructor: Dr. Laura Gross
Due date: Wednesday, November 14, 2007

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1. Consider the spring-mass system modeled by $my'' + \gamma y' + ky = 0$, where $y(t)$ is the vertical displacement of a mass $m > 0$ at time t; $k > 0$ is the spring constant; and $\gamma > 0$ is the damping.

   (a) Write the ordinary differential equation as a system of equations in the variables u and v. Write the system in matrix form.

   (b) What condition must the problem parameters satisfy in order for the origin to be a spiral point in the phase plane? This is the "under-damped" case. Classify the critical point $(0, 0)$ as to stability in this case.

   (c) What condition must the problem parameters satisfy in order for the system to have equal eigenvalues? This is the "critically damped" case. Classify the critical point $(0, 0)$ as to type and stability in this case.

   (d) What condition must the problem parameters satisfy in order for the system to have two distinct eigenvalues? This is the "overdamped" case. Classify the critical point $(0, 0)$ as to type and stability in this case.

   (e) Draw three phase portraits corresponding to the three parameter regimes you identified above.

   (f) Suppose the system is undamped ($\gamma = 0$). Find the eigenvalues in this case, classify the critical point $(0, 0)$ as to type and stability, draw a phase portrait, and interpret the phase portrait physically.

2. Consider the system studied in class $x' = 3x - 2y$, $y' = 4x - y$.

   (a) Write two complex-valued solutions $\mathbf{x}^{(1)} = e^{\lambda t}\mathbf{X}$ and $\mathbf{x}^{(2)} = e^{\bar{\lambda} t}\bar{\mathbf{X}}$. Write these in their real and imaginary parts (using rules of exponents, Euler's formula, and vector arithmetic). Note one is the conjugate of the other.

   (b) Because the problem is linear homogeneous, the superposition principle holds. Write down the solution $\mathbf{x}^{(3)} = \frac{1}{2}\mathbf{x}^{(1)} + \frac{1}{2}\mathbf{x}^{(2)}$. Write down the solution $\mathbf{x}^{(4)} = \frac{1}{2i}\mathbf{x}^{(1)} - \frac{1}{2i}\mathbf{x}^{(2)}$. Use these two linearly independent real-valued solutions $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$ to write down a real-valued general solution to the system. (It's the same one we found in class.) (OVER)

3. Consider the system $x' = x\left(\frac{3}{2} - x - \frac{1}{2}y\right)$, $y' = y\left(2 - y - \frac{3}{4}x\right)$.

   (a) Find the equilibrium solutions (critical points).

   (b) The system is almost linear. For each critical point, find the corresponding linear system. Find the eigenvalues and eigenvectors of each linear system.

   (c) Classify each critical point of the original *nonlinear* system as to type and stability.

   (d) On a single set of x-y axes, sketch some representative trajectories in the neighborhood of each critical point.

   (e) Determine the limiting behavior of x and y as $t \to \infty$.

   (f) Identify the basins of attraction for asymptotically stable critical points.

   (g) Draw a separatrix on your sketch in (3d) if applicable, or explain why no separatrix exists.

   (h) Interpret long-time results in terms of the populations of two competing species: Why is this case called "peaceful coexistence?"