Elementary Differential Equations & Boundary problems, Boyce & DiPrima, 6th ed., p341

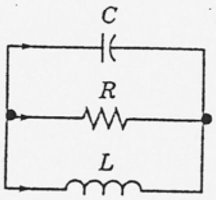

FIGURE 7.1.2 A parallel *LRC* circuit.

Electric Circuits. The theory of electric circuits, such as that shown in Figure 7.1.2, consisting of inductors, resistors, and capacitors, is based on Kirchhoff's laws: (1) The net flow of current through each node (or junction) is zero, (2) the net voltage drop around each closed loop is zero. In addition to Kirchhoff's laws we also have the relation between the current I in amperes through each circuit element and the voltage drop V in volts across the element; namely,

$V = RI,$ R = resistance in ohms;

$C\dfrac{dV}{dt} = I,$ C = capacitance in farads[1];

$L\dfrac{dI}{dt} = V,$ L = inductance in henrys.

Kirchhoff's laws and the current–voltage relation for each circuit element provide a system of algebraic and differential equations from which the voltage and current throughout the circuit can be determined. Problems 18 through 20 illustrate the procedure just described.

18. Consider the circuit shown in Figure 7.1.2. Let I_1, I_2, and I_3 be the current through the capacitor, resistor, and inductor, respectively. Likewise, let V_1, V_2, and V_3 be the corresponding voltage drops. The arrows denote the arbitrarily chosen directions in which currents and voltage drops will be taken to be positive.
    (a) Applying Kirchhoff's second law to the upper loop in the circuit, show that
    $$V_1 - V_2 = 0. \tag{i}$$
    In a similar way, show that
    $$V_2 - V_3 = 0. \tag{ii}$$
    (b) Applying Kirchhoff's first law to either node in the circuit, show that
    $$I_1 + I_2 + I_3 = 0. \tag{iii}$$
    (c) Use the current–voltage relation through each element in the circuit to obtain the equations
    $$CV_1' = I_1, \qquad V_2 = RI_2, \qquad LI_3' = V_3. \tag{iv}$$
    (d) Eliminate V_2, V_3, I_1, and I_2 among Eqs. (i) through (iv) to obtain
    $$CV_1' = -I_3 - \frac{V_1}{R}, \qquad LI_3' = V_1. \tag{v}$$
    Observe that if we omit the subscripts in Eqs. (v), then we have the system (2) of the text.*

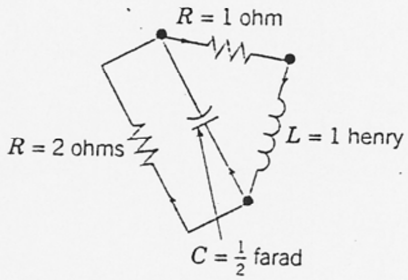

FIGURE 7.1.4 The circuit in Problem 19.

19. Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 18 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations
    $$\frac{dI}{dt} = -I - V, \qquad \frac{dV}{dt} = 2I - V.$$

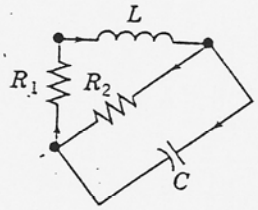

FIGURE 7.1.5 The circuit in Problem 20.

20. Consider the circuit shown in Figure 7.1.5. Use the method outlined in Problem 18 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations
    $$L\frac{dI}{dt} = -R_1 I - V, \qquad C\frac{dV}{dt} = I - \frac{V}{R_2}.$$

* System (2):

$$dI/dt = V/L, \quad dV/dt = -\frac{I}{C} - \frac{V}{RC}$$ (OVER)

Electric Circuits. Problems 32 and 33 are concerned with the electric circuit described by the system of differential equations in Problem 20 of Section 7.1:

$$\frac{d}{dt}\begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} -\dfrac{R_1}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{CR_2} \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}. \qquad \text{(i)}$$

32. (a) Find the general solution of Eq. (i) if $R_1 = 1$ ohm, $R_2 = \frac{3}{5}$ ohm, $L = 2$ henrys, and $C = \frac{2}{3}$ farad.
    (b) Show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ regardless of the initial values $I(0)$ and $V(0)$.
33. Consider the preceding system of differential equations (i).
    (a) Find a condition on R_1, R_2, C, and L that must be satisfied if the eigenvalues of the coefficient matrix are to be real and different.
    (b) If the condition found in part (a) is satisfied, show that both eigenvalues are negative. Then show that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ regardless of the initial conditions.
    *(c) If the condition found in part (a) is not satisfied, then the eigenvalues are either complex or repeated. Do you think that $I(t) \to 0$ and $V(t) \to 0$ as $t \to \infty$ in these cases as well?
    Hint: In part (c) one approach is to change the system (i) into a single second order equation. We also discuss complex and repeated eigenvalues in Sections 7.6 and 7.7.