SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS AND CLASSIFICATION OF THE ORIGIN

3450:438/538-001 Advanced Engineering Math I
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Consider the system $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, A is the matrix of constants $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

and $\det(A) \neq 0$.

Find the eigenvalues and eigenvectors of A. Then the critical point at the origin can be classified as follows:

1.	$\lambda_1 < \lambda_2 < 0$	Improper Node	Asymptotically Stable
2.	$\lambda_1 > \lambda_2 > 0$	Improper Node	Unstable
3.	$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable
4.	$\lambda = \alpha \pm i\beta,\ \alpha < 0$	Spiral Point	Asymptotically Stable
5.	$\lambda = \alpha \pm i\beta,\ \alpha > 0$	Spiral Point	Unstable
6.	$\lambda = \alpha \pm i\beta,\ \alpha = 0$	Center	Stable
7.	$\lambda_1 = \lambda_2 < 0$, e-vtrs indep.	Proper Node	Asymptotically Stable
8.	$\lambda_1 = \lambda_2 > 0$, e-vtrs indep.	Proper Node	Unstable
9.	$\lambda_1 = \lambda_2 < 0$, e-vtrs dep.	Improper Node	Asymptotically Stable
10.	$\lambda_1 = \lambda_2 > 0$, e-vtrs dep.	Improper Node	Unstable

In cases 1, 2, 3, 7, 8, the general solution is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{E^{(1)}} + c_2 e^{\lambda_2 t}\mathbf{E^{(2)}}.$$

In cases 4, 5, 6, the eigenvalue-eigenvector pairs are $(\alpha \pm i\beta, \mathbf{u} \pm i\mathbf{v})$. The real-valued general solution is

$$\mathbf{x}(t) = e^{\alpha t}[c_1(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) + c_2(\mathbf{u}\sin\beta t + \mathbf{v}\cos\beta t)].$$

In cases 9 and 10, the only eigenvalue-eigenvector pair is $(\lambda, [E_1\ E_2]^T)$. The general solution is

$$\mathbf{x}(t) = e^{\lambda t}\left(c_1\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + c_2\left(t\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} 0 \\ E_1/a_{12} \end{bmatrix}\right)\right).$$