

1. Consider the equation  $y''(x) + m^2 y(x) = 0$ . Write the ordinary differential equation as a system of equations in the variables  $u$  and  $v$ .

$$\left. \begin{array}{l} u = y \\ v = y' \end{array} \right\} \begin{array}{l} u' = v \\ v' = -m^2 u \end{array}$$

10 pts

2.  $C[-L, L]$  is a vector space over  $\mathbb{R}$ , where  $L$  is a fixed constant. Let  $f(x), g(x) \in C[-L, L]$ , and let

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^L f(x)g(x) dx. \quad (1)$$

Let

$$S = \left\{ f(x) \mid f(x) = \frac{1}{\sqrt{2}} \text{ or } \sin\left(\frac{n\pi x}{L}\right) \text{ or } \cos\left(\frac{n\pi x}{L}\right), n = 1, 2, \dots \right\}.$$

Write down all the integral formulas that must hold if  $S$  is an orthonormal set in  $C[-L, L]$  with respect to the inner product (1).

10 pts

$$\frac{1}{L} \int_{-L}^L \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} dx = 1, \quad \frac{1}{L} \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx = 1,$$

$$\frac{1}{L} \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx = 1.$$

$$\frac{1}{L} \int_{-L}^L \frac{1}{\sqrt{2}} \sin \frac{n\pi x}{L} dx = 0$$

$$n = 1, 2, 3, \dots$$

$$m = 1, 2, 3, \dots$$

$$\frac{1}{L} \int_{-L}^L \frac{1}{\sqrt{2}} \cos \frac{n\pi x}{L} dx = 0$$

$$\frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0$$

$$\frac{1}{L} \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0, \quad n \neq m$$

$$\frac{1}{L} \int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0, \quad n \neq m \quad (\text{OVER})$$

Laura Gross

3. Consider the ODE  $L(y) = f$ , where  $L(y) = y''$ , on  $-a < x < a$ , subject to boundary conditions  $y'(-a) = 0$  and  $y'(a) = 0$ .  ~~$A$  is a constant (scalar) parameter.~~

Find the adjoint operator  $L^*$  and appropriate boundary conditions for the adjoint problem. **Draw a box around your answer.** By definition we need  $\langle L(y), w \rangle = \langle y, L^*(w) \rangle$ , where we use the inner product  $\langle u, v \rangle = \int_{-a}^a u(x)v(x) dx$ .

$$\begin{aligned} \langle L(y), w \rangle &= \int_{-a}^a y'' \underbrace{w}_{u} dx && \boxed{10 \text{ pts}} \\ &= y'w \Big|_{-a}^a - \int_{-a}^a y' \underbrace{w'}_u dx \\ &= \underbrace{y'(a)w(a)}_{=0} - \underbrace{y'(-a)w(-a)}_{=0} - \left[ w'y \Big|_{-a}^a - \int_{-a}^a yw'' dx \right] \\ &= -w'(a)y(a) + w'(-a)y(a) + \int_{-a}^a yw'' dx \\ &= \langle y, L^*(w) \rangle \text{ if } \boxed{L^*(w) = w'' \ \& \ w'(a) = w'(-a) = 0} \end{aligned}$$

4. Consider the system  $x' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} x$ .

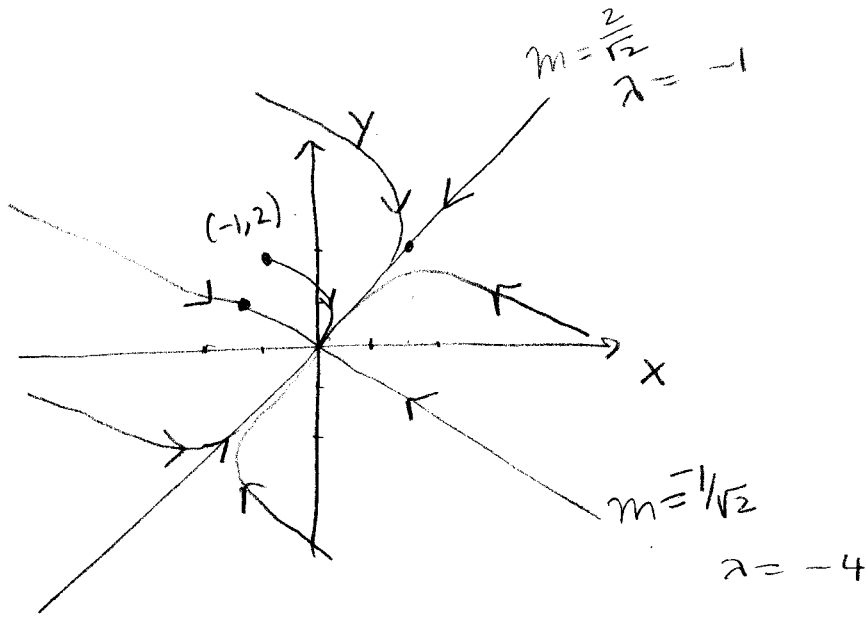
(a) Verify the solution  $x = c_1 e^{-4t} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix}$  by substituting it into the system.

$$\begin{aligned} Ax &= \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} \begin{bmatrix} -c_1 e^{-4t} \sqrt{2} + c_2 e^{-t} \sqrt{2} \\ c_1 e^{-4t} + c_2 e^{-t} 2 \end{bmatrix} && \boxed{10 \text{ pts}} \\ &= \begin{bmatrix} (3\sqrt{2} + \sqrt{2})c_1 e^{-4t} + c_2(-3\sqrt{2} + 2\sqrt{2})e^{-t} \\ (-2 - 2)c_1 e^{-4t} + c_2(2 - 4)e^{-t} \end{bmatrix} \\ \underline{x}' &= \begin{bmatrix} 4\sqrt{2}c_1 e^{-4t} - \sqrt{2}c_2 e^{-t} \\ -4c_1 e^{-4t} - 2c_2 e^{-t} \end{bmatrix} \end{aligned}$$

← ⊖ ✓

(b) Draw a sketch of a representative set of trajectories in the phase plane.

10 pts



(c) Simply using your answer to (4b), draw an approximate sketch of the solution to the differential equation subject to the initial condition  $x_1(0) = -1, x_2(0) = 2$  on your graph above.

10 pts

(d) Classify the critical point  $(0, 0)$  as to type and stability.

10 pts

asymptotically stable  
improper node

Yama Kross

5. Let  $S$  be the subspace of the vector space  $\mathbb{R}^3$ , where  $S$  consists of all elements of the form  $(a+b, c+a, 0)$ .

(a) Determine a basis for  $S$ , and state the dimension of  $S = 2$ .

10 pts

$$\{(1, 0, 0), (0, 1, 0)\} = B$$

(b) Prove that the set you identified in (5a) is a basis.

10 pts

$$B \subseteq S$$

$$(i) \quad \alpha_1(1, 0, 0) + \alpha_2(0, 1, 0) = (0, 0, 0) \rightarrow \alpha_1 = \alpha_2 = 0$$

$B$  is linearly indep.

(ii) Let  $x$  be an element of  $S$ ,

$$x = (a+b, c+a, 0) = (a+b)(1, 0, 0) + (c+a)(0, 1, 0)$$

So  $B$  spans  $S$ .

So  $B$  is a basis.

6. Let  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  be elements of  $\mathbb{C}^2$ . Define the function on  $\mathbb{C}^2$ :  $\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$ . Prove that the inner-product property  $\langle a+b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$  is satisfied for this function.

$$\langle a+b, c \rangle = (a_1+b_1)\bar{c}_1 + (a_2+b_2)\bar{c}_2$$

10 pts

$$= (a_1\bar{c}_1 + a_2\bar{c}_2) + (b_1\bar{c}_1 + b_2\bar{c}_2)$$

$$= \langle a, c \rangle + \langle b, c \rangle \quad \checkmark$$