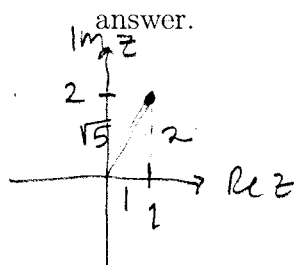


Name: _____

Laura Gross

1. Write $(1 + 2i)^{-\pi}$ in a form $x + iy$. (Finding the principal value is sufficient.) Give an exact



$$(\sqrt{5} e^{i \tan^{-1} 2})^{-\pi}$$

11 pts

$$= \sqrt{5}^{-\pi} e^{-i\pi \tan^{-1} 2}$$

$$= \sqrt{5}^{-\pi} [\cos(\pi \tan^{-1} 2) - i \sin(\pi \tan^{-1} 2)]$$

$$= \sqrt{5}^{-\pi} [\cancel{\cos \pi} \cos(\tan^{-1} 2) - \cancel{\sin \pi} \sin(\tan^{-1} 2) +$$

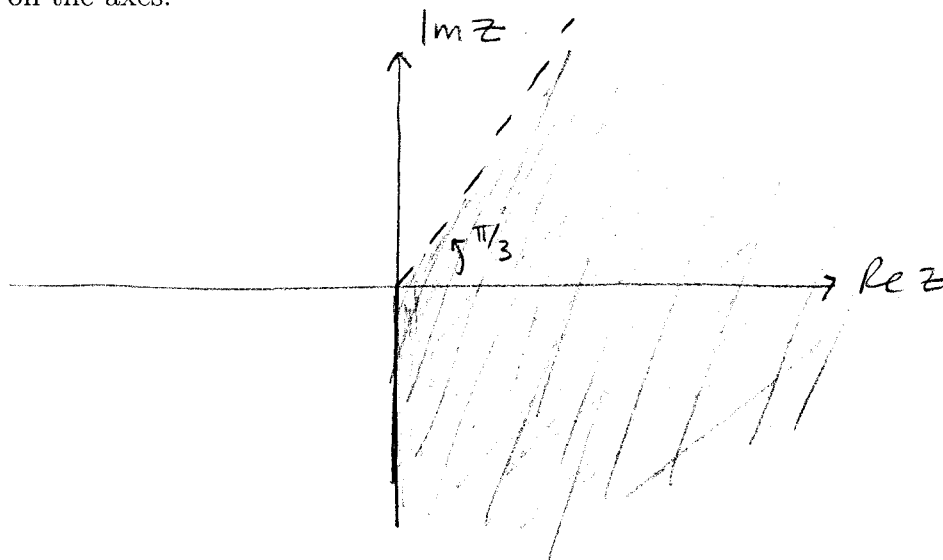
$$- i (\cancel{\sin \pi} \cos(\tan^{-1} 2) + \cancel{\cos \pi} \sin(\tan^{-1} 2))] = \sqrt{5}^{-\pi} [-\frac{1}{\sqrt{5}} + i \frac{2}{\sqrt{5}}] = \sqrt{5}^{-\pi-1} (-1 + 2i)$$

2. Prove $\bar{\bar{z}} = z$ for all complex numbers z .

$$\overline{\overline{z}} = \overline{(x+iy)} = \overline{x-iy} = x+iy = z \quad \checkmark$$

11 pts

3. Sketch the set of points in the complex plane that satisfies $-\pi/2 \leq \arg(z) < \pi/3$. Indicate the scale on the axes.



11 pts

(OVER)

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4. Use the definition of the complex sine and cosine functions to prove the sum identity for cosine. Namely, show $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$.

11 pts

$$\begin{aligned}
 & \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
 = & \frac{e^{iz_1} + e^{-iz_1}}{2} \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right) + \frac{e^{iz_1} - e^{-iz_1}}{2i} \left(\frac{e^{iz_2} - e^{-iz_2}}{2i} \right) \\
 = & \frac{1}{4} \left[e^{i(z_1+z_2)} + e^{i(z_2-z_1)} + e^{i(z_1-z_2)} + e^{-i(z_1+z_2)} \right. \\
 & \left. + e^{i(z_1+z_2)} - e^{i(z_2-z_1)} - e^{i(z_1-z_2)} + e^{-i(z_1+z_2)} \right] \\
 = & \frac{1}{2} \left[e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} \right] = \cos(z_1+z_2) \checkmark
 \end{aligned}$$

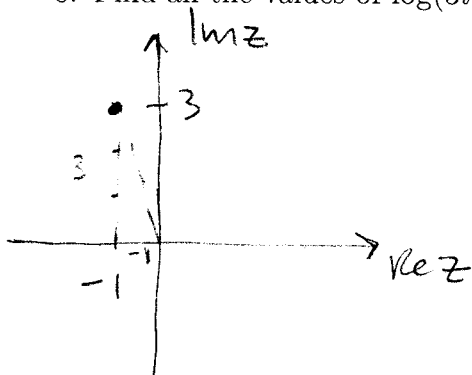
5. For two complex numbers z_1 and z_2 , prove that if $z_1 = z_2 + k2\pi i$, $k \in \mathbb{Z}$, then $e^{z_1} = e^{z_2}$.

$$\begin{aligned}
 e^{z_1} &= e^{z_2 + k2\pi i} = e^{z_2} e^{k2\pi i}, \quad k \in \mathbb{Z} \\
 &= e^{z_2} \cdot 1 \\
 &= e^{z_2} \checkmark
 \end{aligned}$$

11 pts

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6. Find all the values of $\log(3i - 1)$.



11 pts

$$\begin{aligned} \log(3i - 1) &= \log(\sqrt{10} e^{i[\tan^{-1}(-3) + \pi + k2\pi]}) \\ &= \ln \sqrt{10} + i[\tan^{-1}(-3) + \pi(2k+1)] \end{aligned}$$

$$k \in \mathbb{Z}$$

7. Show that $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$, where $z = x + iy$, is differentiable.

One way: $u(x, y) = x^3 - 3xy^2$
 $v(x, y) = 3x^2y - y^3$

11 pts

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -6xy = -\frac{\partial v}{\partial x} \end{aligned} \right\} \text{C-R eqns are satisfied, \&}$$

$u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are continuous,

So $f(z)$ is differentiable.

(OVER)

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8. Let $L(u) = \frac{du}{dt} + u^2$.

(a) Prove or disprove that L is a linear operator.

11 pts

$$L(\alpha u) = \frac{d}{dt}(\alpha u) + (\alpha u)^2$$

$$= \alpha \frac{du}{dt} + \alpha^2 u^2$$

$$= \alpha \left(\frac{du}{dt} + \alpha u^2 \right)$$

$$\alpha L(u) = \alpha \left(\frac{du}{dt} + u^2 \right)$$

⊆ in general.

So L is not a linear operator.(b) Let S be the set of continuously differentiable solutions to the equation $L(u) = 0$. Prove or disprove that S is a subspace of the vector space of continuously differentiable functions. Do not solve the differential equation.

11 pts

 S is a subset of the v.s. of continuously differentiable fncsBut S is not closed, e.g.

Let $u \in S \Leftrightarrow L(u) = 0 \Leftrightarrow \frac{du}{dt} + u^2 = 0$ (*)

Let $\alpha \in \mathbb{R}$.

$$\alpha u \stackrel{?}{\in} S \Leftrightarrow L(\alpha u) \stackrel{?}{=} 0 \Leftrightarrow \frac{d}{dt}(\alpha u) + (\alpha u)^2 \stackrel{?}{=} 0$$

$$\frac{d}{dt}(\alpha u) + (\alpha u)^2 = \alpha \frac{du}{dt} + \alpha^2 u^2 \quad \left. \begin{array}{l} \text{Can't conclude} \\ \text{this is 0 from} \end{array} \right\} \text{the info (*)}$$

$$= \alpha(-u^2) + \alpha^2 u^2 = \alpha u^2(\alpha - 1)$$

 $\neq 0$ in general. S is not closed under scalar mult, so S