AEM I
F2007
Gross

HW 9 – A Sol'n

(1). $A_1 = x_1 + i x_2,\quad A_2 = x^3 + i x^4 \rightarrow$

$$x_1' + i x_2' = -iK_1(x_1 + i x_2) + i(x_1 - i x_2)(x_3 + i x_4) +$$
$$- \varepsilon\alpha(x_1 + i x_2)$$

$$x_3' + i x_4' = -i\varepsilon K_2(x_3 + i x_4) + \frac{i}{2}(x_1 + i x_2)^2 - \varepsilon\alpha(x_3 + i x_4)$$
$$+ \varepsilon\gamma_2$$

Equate real & imag parts $\rightarrow$

$$x_1' = K_1 x_2 - (x_1 x_4 - x_2 x_3) - \varepsilon\alpha x_1$$

$$x_2' = -K_1 x_1 + x_1 x_3 + x_2 x_4 - \varepsilon\alpha x_2$$

$$x_3' = \varepsilon K_2 x_4 - \frac{1}{\cancel{2}}\cancel{2} x_1 x_2 - \varepsilon\alpha x_3 + \varepsilon\gamma_2$$

$$x_4' = -\varepsilon K_2 x_3 + \frac{1}{2}(x_1^2 - x_2^2) - \varepsilon\alpha x_4$$

(2). $ay'' + by' + cy = 0$

(a) $y = c_1 e^{r_+ x} + c_2 e^{r_- x}, \quad r_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(b) $u = y,\ v = y' \Rightarrow u' = v$

$$v' = \frac{-bv - cu}{a}$$

(c) $$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

(d) $(0 + \lambda)\left(+\frac{b}{a} + \lambda\right) + \frac{c}{a} = 0 \Rightarrow$

$$\lambda(b + \lambda a) + c = 0$$

$$\lambda^2 a + \lambda b + c = 0 \Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad ✓$$

(2e) $a=1, b=1, c=-6 \Rightarrow$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{25}}{2} = 2, -3.$$

$y = c_1 e^{2x} + c_2 e^{-3x}$ from (a)

(d) $\Rightarrow \lambda = 2, -3$ for $A = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}$.

$\lambda = 2$: $(A-\lambda I)\underline{X} = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. ✓ ☺

$\lambda = -3$: $(A-\lambda I)\underline{X} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

$\begin{bmatrix} u \\ v \end{bmatrix} = k_1 e^{2x} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + k_2 e^{-3x} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

$u = k_1 e^{2x} + k_2 e^{-3x} = y$

Note $v = k_1 2 e^{2x} + k_2(-3)e^{-3x} = y'$ ✓

(3)(a) $x_2 = x_1' + 2x_1$

$(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$.

$x_1'' + 4x_1' + 3x_1 = 0$, $r^2 + 4r + 3 = 0 \to (r+3)(r+1) = 0$

$r = -3, -1 \Rightarrow \underline{x_1 = c_1 e^{-3t} + c_2 e^{-t}}$

$x_2 = x_1' + 2x_1 = -3c_1 e^{-3t} - c_2 e^{-t}$
$\quad + 2c_1 e^{-3t} + 2c_2 e^{-t}$

$\underline{x_2 = -c_1 e^{-3t} + c_2 e^{-t}}$

(3b). $x_1(0) = c_1 + c_2 = 2$

$x_2(0) = -c_1 + c_2 = 3$

$2c_2 = 5 \rightarrow c_2 = 5/2$

$c_1 = 2 - c_2 = 2 - \frac{5}{2} = -1/2$

$$x_1 = -\frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t}$$

$$x_2 = \frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t}$$

(c). $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (2+\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = 0$$

$\rightarrow \lambda = -1, -3.$

$\lambda = -1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left.\begin{matrix} k_1 + k_2 = 2 \\ k_1 - k_2 = 3 \end{matrix}\right\} \rightarrow \begin{matrix} k_1 = 5/2 \\ k_2 = -1/2 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{5}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left.\begin{matrix} x_1 = -\frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t} \\ x_2 = \frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t} \end{matrix}\right\} \text{as in (b)} ✓$$

(4) $\begin{vmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{vmatrix} = +\lambda^2(4-\lambda) + (-1)(4-\lambda)$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$ Note $\lambda = 1$ is a sol'n

$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \iff \lambda = 4, -1$

$$\begin{array}{r|l} \lambda - 1 & \lambda^3 - 4\lambda^2 - \lambda + 4 \\ & \lambda^3 - \lambda^2 \\ \hline & -3\lambda^2 - \lambda + 4 \\ & -3\lambda^2 + 3\lambda \\ \hline & -4\lambda + 4 \end{array}$$

$\lambda = 1$: $\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix} \underline{X} = 0$ $\xrightarrow[E_3 - E_1 \to E_3]{E_2 + 2E_1 \to E_2}$ $\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right]$

$\underline{X} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$\lambda = 4$: $\begin{bmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{bmatrix} \underline{X} = \underline{0}$ $\underline{X} = \begin{bmatrix} -1 \\ -1/2 \\ 4 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{bmatrix} \underline{X} = \underline{0}$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$, $\underline{X} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\boxed{\underline{X} = c_1 e^{t} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$\underline{X}(0)$: $\left[\begin{array}{ccc|c} -1 & -2 & 1 & 7 \\ -2 & -1 & -2 & 5 \\ 1 & 8 & 1 & 5 \end{array}\right] \xrightarrow[R_3 + R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} +1 & 2 & -1 & -7 \\ 0 & 3 & -4 & -9 \\ 0 & 6 & 2 & 12 \end{array}\right]$ $\boxed{\begin{array}{l} c_1 = -6 \\ c_2 = 1 \\ c_3 = 3 \end{array}}$

$\left[\begin{array}{ccc|c} 0 & 0 & 10 & 30 \end{array}\right]$ $\frac{-9 + 12}{3}$ $-7 - 2 + 3$