HW 8 - A Sol'n

AEM I
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(1) $y'' + y = 0$, $y(0) = 0$, $y(1) = 0$

(a) $y = e^{rx} \to r^2 + 1 = 0 \to r = \pm i \to y = c_1 \cos x + c_2 \sin x$.

$y(0) = c_1 = 0 \to y = c_2 \sin x$

$y(1) = c_2 \sin(1) = 0 \to c_2 = 0 \to y(x) \equiv 0$

(b) Dimension of v.s. of sol'ns is 0.

(2) $y'' + \pi^2 y = 0$, $y(0) = 0$, $y(1) = 0$

(a) $y = e^{rx} \to r^2 + \pi^2 = 0 \to r = \pm \pi i \to y = c_1 \cos \pi x + c_2 \sin \pi x$

$y(0) = c_1 = 0 \to y = c_2 \sin \pi x$

$y(1) = c_2 \sin \pi = 0$ ✓ $\to y(x) = c_2 \sin \pi x$

(b) $B = \{\sin \pi x\}$

(3) $y'' + \pi^2 y = 0$, $y(0) + y(1) = 0$, $y'(0) + y'(1) = 0$.

(a) $y = c_1 \cos \pi x + c_2 \sin \pi x$ as in (2)

$y(0) + y(1) = c_1 + c_1(-1) + c_2 \sin \pi = 0$ ✓

$y'(x) = -c_1 \pi \sin \pi x + c_2 \pi \cos \pi x$.

$y'(0) + y'(1) = c_2 \pi + c_2 \pi(-1) = 0$ ✓

$y(x) = c_1 \cos \pi x + c_2 \sin \pi x$

(b) $B = \{\cos \pi x, \sin \pi x\}$.

(4) $y'' + y = \sin x$

① $y_h = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}$

② $y_p = u\, y_1 + v\, y_2$

③ $u' = \dfrac{-y_2 f}{W}$ , $v' = \dfrac{y_1 f}{W}$ , $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = +1$

$= -\sin^2 x$ $\qquad = +\cos x \sin x$

$u = -\int \sin^2 x\, dx = -\int \dfrac{1-\cos 2x}{2}\, dx = -\frac{1}{2} x + \dfrac{\sin 2x}{4}$

$v = +\int \cos x \sin x\, dx = -\dfrac{\cos^2 x}{2}$

$y_p = -\frac{1}{2} x \cos x + \frac{1}{4} \sin 2x \cos x - \frac{1}{2}\cos^2 x \sin x$

$= -\frac{1}{2} x \cos x + \frac{1}{2} \sin x \cos^2 x - \frac{1}{2}\cos^2 x \sin x$

$= -\frac{1}{2} x \cos x$

(a) $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$

(b) $y(0) = c_1 = 0 \rightarrow \boxed{y = c_2 \sin x - \frac{1}{2} x \cos x}$

$y(\pi) = +\frac{1}{2}\pi$ ✓

(c) $y(0) = c_1 = 0$

$y(\pi/2) = c_2 = 1 \rightarrow \boxed{y = \sin x - \frac{1}{2} x \cos x}$

(d) $y(0) = c_1 = 0$

$y(\pi) = +\frac{1}{2}\pi = 1$ contradiction! $\boxed{\text{No sol}^{\underline{n}}}$

(e) (b) is underdetermined

(c) is well-posed

(d) is overdetermined.

(5) (a) $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \xrightarrow[R_3 - R_2 \to R_2]{} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$

every column is a pivot column.

→ There are no free variables $x_i$ in $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

→ $\mathcal{N}(A) = \{\underline{0}\}$.

→ $L$ is 1-1.

(b) $A^* = \begin{array}{cc} & \begin{array}{ccc} -2 & 0 & -2 \end{array} \\ -2 & \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[R_2 - 2R_1 \to R_2]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$

→ Column 3 is not a pivot column.

→ The variable $x_3$ in $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is free.

→ $\mathcal{N}(A^*) \neq \{\underline{0}\}$

→ $L$ is not onto.

(c). $L$ is not both 1-1 & onto, so it's not invertible.

(6) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(a). Not every column has a pivot → $\mathcal{N}(A) \neq \{\underline{0}\}$

→ $L$ is not 1-1.

(b) $A^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix}$. Every column is a pivot column.

→ $\mathcal{N}(A^*) = \{\underline{0}\}$ → $L$ is onto.

(c). $L$ is not both 1-1 & onto, so it's not invertible.

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(7) $A = \begin{bmatrix} -i & 2+2i \\ -1+i & 3 \\ -3i & 1 \end{bmatrix}$, $A^* = \begin{bmatrix} i & -1-i & 3i \\ 2-2i & 3 & 1 \end{bmatrix}$.

$$\langle A\underline{x}, \underline{y} \rangle = \left\langle \begin{bmatrix} -i & 2+2i \\ -1+i & 3 \\ -3i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} -ix_1 + (2+2i)x_2 \\ (-1+i)x_1 + 3x_2 \\ -3ix_1 + x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle$$

$$= -ix_1\bar{y}_1 + (2+2i)x_2\bar{y}_1 + (-1+i)x_1\bar{y}_2 + 3x_2\bar{y}_2 + -3ix_1\bar{y}_3 + x_2\bar{y}_3$$

$$\langle \underline{x}, A^*\underline{y} \rangle = \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} +i & -1-i & 3i \\ 2-2i & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} iy_1 + (-1-i)y_2 + 3iy_3 \\ (2-2i)y_1 + 3y_2 + y_3 \end{bmatrix} \right\rangle$$

$$= x_1\overline{(iy_1 + (-1-i)y_2 + 3iy_3)} + x_2\overline{((2-2i)y_1 + 3y_2 + y_3)}$$

$$= -ix_1\bar{y}_1 + (-1+i)x_1\bar{y}_2 + (-3i)x_1\bar{y}_3 + (2+2i)x_2\bar{y}_1 + 3x_2\bar{y}_2 + x_2\bar{y}_3$$

$=$ ✓

(8) $y'' = x^2 + A$, $-a < x < a$, $y'(-a) = y'(a) = 0$

(a) $L(y) = f$, where $L(y) = y''$, & $f = x^2 + A$

(b). $\langle L(y), w \rangle = \int_{-a}^{a} \underbrace{y''}_{dv} \overset{u}{w}\, dx$

$= wy' \Big|_{-a}^{a} - \int_{-a}^{a} y' \overset{u}{w'}\, dx$

$= w(a)\cancel{y'(a)}^{0} - w(-a)\cancel{y'(-a)}^{0} - w'y\, dx \Big|_{-a}^{a}$

$+ \int_{-a}^{a} y w''\, dx$

$= \langle y, L^*(w) \rangle$,

where $L^*(w) = w''$ if $w'(a) = 0$, $w'(-a) = 0$

(c) $w'' = 0$, $w'(a) = 0$, $w'(-a) = 0$.

$w' = c_1 \rightarrow w = c_1 x + c_2$

$w' = c_1$

$w'(a) = 0 \rightarrow c_1 = 0$.

$w = c_2$

$w'(-a) = 0$ ✓

$w = c_2 \rightarrow \mathcal{N}(L^*) = \{c \mid c \text{ is const}\}$

(d). $\mathcal{N}(L) = \{c\}$ also, as the problem is self-adjoint.

(e) $\mathcal{N}(L) \neq \{0\}$, & $\mathcal{N}(L^*) \neq \{0\}$, so. Fredholm Alt. (II) holds.