HW 7 - A sol'n

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(1) (i) Let $A, B, C \in M_{22}$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(ii) Let $\alpha \in \mathbb{R}$.

Check 4 properties:

[1] $\langle A+B, C\rangle \stackrel{?}{=} \langle A, C\rangle + \langle B, C\rangle$

$$\langle A+B, C\rangle = \left\langle \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right\rangle$$

$$= (a_{11}+b_{11})c_{11} + (a_{12}+b_{12})c_{12} + (a_{21}+b_{21})c_{21} + (a_{22}+b_{22})c_{22}$$

$$= a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22} + b_{11}c_{11} + b_{12}c_{12} + b_{21}c_{21} + b_{22}c_{22}$$

$$= \langle A, C\rangle + \langle B, C\rangle \ \checkmark$$

[2] $\langle \alpha A, B\rangle \stackrel{?}{=} \alpha\langle A, B\rangle$

$$\langle \alpha A, B\rangle = \alpha a_{11}b_{11} + \alpha a_{12}b_{12} + \alpha a_{21}b_{21} + \alpha a_{22}b_{22}$$

$$= \alpha\langle A, B\rangle \ \checkmark$$

[3] $\langle B, A\rangle \stackrel{?}{=} \overline{\langle A, B\rangle}$

$$\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

$= \overline{\langle A, B\rangle}$ because $\langle A, B\rangle$ is real.

$$= \langle B, A\rangle \ \checkmark$$

[4] $\langle A, A\rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 > 0$ if $A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

(1b) $\left\langle \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 9 & 0 \end{bmatrix} \right\rangle = 10 - 6 + 0 + 0 = \boxed{4}$

(4). (a) $\langle \underline{x}, \underline{y} \rangle \in \mathbb{R}$ ✓

Check 4 properties: $\left\{ \begin{array}{l} \text{let } \underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^2 \\ \text{let } \alpha \in \mathbb{R} \end{array} \right.$

① $\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle$

$= (x_1 + y_1) z_1 + 4 (x_2 + y_2) z_2$

$= (x_1 z_1 + 4 x_2 z_2) + (y_1 z_1 + 4 y_2 z_2)$

$= \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$ ✓

② $\langle \alpha \underline{x}, \underline{y} \rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2) \rangle$

$= \alpha x_1 y_1 + 4 \alpha x_2 y_2 = \alpha (x_1 y_1 + 4 x_2 y_2)$

$= \alpha \langle \underline{x}, \underline{y} \rangle$ ✓

③ $\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + 4 x_2 y_2 = y_1 x_1 + 4 y_2 x_2$

$= \langle \underline{y}, \underline{x} \rangle$ real

$= \overline{\langle \underline{y}, \underline{x} \rangle}$ ✓

④ $\langle \underline{x}, \underline{x} \rangle = x_1^2 + 4 x_2^2 > 0$ if $\underline{x} \neq \underline{0}$

(b). $B = \{ (-8, 2), (1, 1) \}$

① $B \subseteq \mathbb{R}^2$

② Neither element is a multiple of the other, so B is linearly indep.

③ An arbitrary element $(x, y) \in \mathbb{R}^2$ can be written as a linear comb'n of $(-8, 2)$ & $(1, 1)$:

$(x,y) = \alpha_1(-8,2) + \alpha_2(1,1) \Rightarrow$ HW 7.3

$$x = -8\alpha_1 + \alpha_2$$
$$y = 2\alpha_1 + \alpha_2$$

$$\left[\begin{array}{cc|c} -8 & 1 & x \\ 2 & 1 & y \end{array}\right] \to \begin{array}{c} 4\times \\ \end{array}\left[\begin{array}{cc|c} 2 & 1 & y \\ -8 & 1 & x \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 2 & 1 & y \\ 0 & 5 & 4y+x \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & y \\ 0 & 1 & \frac{4y+x}{5} \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & \frac{y-x}{5} \\ 0 & 1 & \frac{4y+x}{5} \end{array}\right]$$

$$\alpha_1 = \frac{y-x}{10}, \quad \alpha_2 = \frac{4y+x}{5}$$

~~(1)–(3)~~ $\Rightarrow$ B is a basis

(4) $\langle (-8,2),(1,1) \rangle = -8 + 4(2) = 0$ ✓

The elements are orthogonal w.r.t. the i.p. (1)

(1)–(4) $\Rightarrow$ B is an orthogonal basis.

For ex $\langle (1,1),(1,1) \rangle = 1 + 4 \neq 1$, so B is not an orthonormal basis.

(c) $(-6,4) = \frac{4+6}{10}(-8,2) + \frac{16-6}{5}(1,1)$ by (1b(3))

$$= (-8,2) + 2(1,1)$$

(5) (a) $\langle f,g \rangle \in \mathbb{R}$ ✓ $\begin{cases} \text{Let } f,g,h \in C[-L,L]. \\ \text{Let } \alpha \in \mathbb{R}. \end{cases}$

Check 4 properties:

(1) $\langle f+g, h \rangle = \frac{1}{L}\int_{-L}^{L} (f+g)(x)\,h(x)\,dx =$

$$= \frac{1}{L}\int_{-L}^{L} [f(x)h(x) + g(x)h(x)]\,dx =$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)h(x)\,dx + \frac{1}{L}\int_{-L}^{L} g(x)h(x)\,dx$$

$$= \langle f,h \rangle + \langle g,h \rangle \checkmark$$

(2) $\langle \alpha f, g \rangle = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) g(x))\,dx =$

$$= \alpha \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx =$$

$$= \alpha \langle f,g \rangle \checkmark$$

③ $\langle f, g \rangle = \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx$

$= \langle g, f \rangle$ real

$= \overline{\langle g, f \rangle}$ ✓

④ $\langle f, f \rangle = \frac{1}{L}\int_{-L}^{L} [f(x)]^2\,dx$ non-negative integrand →

> 0 for $f(x) \not\equiv 0$

(b). All distinct pairs must have i.p. $= 0$:

$$\langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

$= 0$ for $n \neq m$.

$$\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

$= 0$ for $n \neq m$.

$$\langle \sin\frac{n\pi x}{L}, \cos\frac{m\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

$= 0$.

$$\langle \frac{1}{\sqrt{2}}, \sin\frac{n\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \frac{1}{\sqrt{2}}\sin\frac{n\pi x}{L}\,dx$$

$= 0$.

$$\langle \frac{1}{\sqrt{2}}, \cos\frac{n\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \frac{1}{\sqrt{2}}\cos\frac{n\pi x}{L}\,dx = 0$$

Each element must have an inner product of 1 with itself:

$$\langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L} \rangle = \frac{1}{L}\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)dx = 1$$

$$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{1}{L}\int_{-L}^{L} \frac{1}{2}\,dx = 1.$$

Note in $g(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$,

$$a_0 = \left\langle g(x), \tfrac{1}{\sqrt{2}} \right\rangle = \frac{1}{L}\int_{-L}^{L} \frac{1}{\sqrt{2}}\, g(x)\, dx.$$

(6)

$$\left[\begin{array}{ccccc|c} -3 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 4 & 0 \\ 0 & 0 & -3 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - R_2 \to R_1]{} \left[\begin{array}{ccccc|c} -3 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 & 4 & 0 \\ 0 & 0 & -3 & 2 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} -3 & 0 & -2 & 1 & -3 & 0 \\ 0 & 3 & 3 & 0 & 12 & 0 \\ 0 & 0 & -3 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_2 + R_3 \to R_2]{} \left[\begin{array}{ccccc|c} -3 & 0 & -2 & 1 & -3 & 0 \\ 0 & 3 & 0 & 2 & 13 & 0 \\ 0 & 0 & -3 & 2 & 1 & 0 \end{array}\right]$$

$-R_1/3 \to R_1$, $R_2/3 \to R_2$, $R_3/-3 \to R_3$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2/3 & -1/3 & 1 & 0 \\ 0 & 1 & 0 & 2/3 & 13/3 & 0 \\ 0 & 0 & 1 & -2/3 & -1/3 & 0 \end{array}\right] \xrightarrow[R_1 - \frac{2}{3}R_3 \to R_1]{} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1/9 & 11/9 & 0 \\ 0 & 1 & 0 & 2/3 & 13/3 & 0 \\ 0 & 0 & 1 & -2/3 & -1/3 & 0 \end{array}\right].$$

$0 \quad 0 \quad -2/3 \quad 4/9 \quad 2/9 \quad 0$

(b) Rank(A) = # pivots = 3.

(c)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9}t - \frac{11}{9}s \\ -\frac{2}{3}t - \frac{13}{3}s \\ \frac{2}{3}t + \frac{1}{3}s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -1/9 \\ -2/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -11/9 \\ -13/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$$

(d) $B = \left\{ \begin{bmatrix} -1/9 \\ -2/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11/9 \\ -13/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$. (e) $\dim N(A) = 2$.

(7)

(a)
$$\begin{array}{cc} & \begin{array}{cccc} -2 & -4 & -6 & -2b_1 \end{array} \\ -2 & \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \end{array}\right] \qquad \boxed{b_2 - 2b_1 = 0}$$

(b) $\mathrm{Col}(A) \subseteq \mathbb{R}^3$ ✓ Let $\underline{b}, \underline{c} \in \mathrm{Col}(A)$

(so $\underline{b} = \beta_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + \beta_3 \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$ for some $\beta_i, \gamma_i \in \mathbb{R}$)

$\underline{c} = \gamma_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \gamma_2 \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} + \gamma_3 \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$. let $\alpha, \beta \in \mathbb{R}$.

① $\underline{b} + \underline{c} \overset{?}{\in} Col(A)$

$$\underline{b} + \underline{c} = (\beta_1 + \gamma_1)\begin{bmatrix}1\\2\\2\end{bmatrix} + (\beta_2 + \gamma_2)\begin{bmatrix}2\\4\\5\end{bmatrix} + (\beta_3 + \gamma_3)\begin{bmatrix}3\\6\\7\end{bmatrix}$$

$\in Col(A)$ by def. ✓

② $\alpha \underline{b} \overset{?}{\in} Col(A)$.

$$\alpha \underline{b} = \alpha\beta_1\begin{bmatrix}1\\2\\2\end{bmatrix} + \alpha\beta_2\begin{bmatrix}2\\4\\5\end{bmatrix} + \alpha\beta_3\begin{bmatrix}3\\6\\7\end{bmatrix}$$

$\in Col(A)$ by def. ✓

(c) $\left\{ \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\4\\5\end{bmatrix}, \begin{bmatrix}3\\6\\7\end{bmatrix} \right\}$ spans $Col(A)$ by def.

Is it linearly indep?

$$\alpha_1\begin{bmatrix}1\\2\\2\end{bmatrix} + \alpha_2\begin{bmatrix}2\\4\\5\end{bmatrix} + \alpha_3\begin{bmatrix}3\\6\\7\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\2 & 4 & 6 & 0\\2 & 5 & 7 & 0\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\0 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{array}\right] \quad (4a).$$

α_3 is free (need not be zero) so the set is not a basis.

$\left\{ \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\4\\5\end{bmatrix} \right\}$ $\subseteq Col(A)$. is linearly indep. & spans $Col(A)$. (Prove these!)

It is a basis.

$\dim(Col(A)) = 2$.

(d). $b_2 - 2b_1 = -2 - 2(-1) = 0$ ✓

(e). (7a) $\rightarrow$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \end{array}\right]$

3 − 2(−1)

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 2 & 2 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{\to R_1 \\ R_2/2 \to R_2}]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -11 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11-t \\ 5-t \\ t \end{bmatrix}$$

check:
$-11-t + 2(5-t) + 3t = -1$ ✓
$2(-11-t) + 4(5-t) + 6t = -2$ ✓
$2(-11-t) + 5(5-t) + 7t = 3$ ✓

(f). $$\underline{x} = \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t}_{\underline{x}_h} + \underbrace{\begin{bmatrix} -11 \\ 5 \\ 0 \end{bmatrix}}_{\underline{x}_p}$$

Note $\dim \mathcal{N}(A) = 1$.

(2). $\langle (2+3i, -1+5i), (1+i, -i) \rangle =$

$= (2+3i)\overline{(1+i)} + (-1+5i)\overline{(-i)}$

$= (2+3i)(1-i) + (-1+5i)i$ in form $x+iy$ is:

$= 2+3-5+i(3-2-1) = 0$

(The vtrs are orthogonal w.r.t. this i.p.).

(3). $\langle 3x-2, x \rangle = \int_0^1 (3x-2)x \, dx =$

$= \int_0^1 (3x^2 - 2x)\, dx = (x^3 - x^2)\Big|_0^1 = 1-1-0+0$

$= 0$ ✓