HW6-A sol$^{\underline{n}}$

AMM I
F2007
Gross

(1) Let u & v satisfy $L(x)=0$;
i.e. $L(u)=0$, $L(v)=0$
show $L(\alpha u+\beta v)=0$

$$L(\alpha u+\beta v) = \alpha L(u)+\beta L(v) \quad \text{because } L \text{ is a linear operator}$$
$$= \alpha\cdot 0+\beta\cdot 0$$
$$= 0 \ \checkmark$$

(2) Superposition doesn't hold for nonlinear equations $T(x)=f$ because in the arguement above we cannot take the step $T(\alpha u+\beta v) = \alpha T(u)+\beta T(v)$ for a nonlinear operator.

(3) Superposition doesn't hold for nonhomog equations $L(x)=f$, $f\neq 0$, because in the arguement in (1), we'd try to say

$$L(\alpha u+\beta v)=f, \text{ but}$$
$$L(\alpha u+\beta v) = \alpha L(u)+\beta L(v) \text{ would} = \alpha f+\beta f \neq f \text{ in general}$$

(4) $L: S_1 \to S_2 \Rightarrow \left\{\begin{array}{l}\text{sol'ns to } L(x)=0\\ \text{are in } S_1\end{array}\right.$

$\Rightarrow S \subseteq S_1$

Let $u, v \in S. \Rightarrow L(u)=0, L(v)=0$.

(i) $u+v \overset{?}{\in} S \iff L(u+v) \overset{?}{=} 0$

$L(u+v) = L(u)+L(v) = 0+0 = 0 \ \checkmark$

(ii) $\alpha u \overset{?}{\in} S \iff L(\alpha u) \overset{?}{=} 0$

$L(\alpha u) = \alpha L(u) = \alpha 0 = 0 \ \checkmark$

(5). $S = \text{Span}(2x+3,\ x-1,\ -x-4)$

(a) $2x+3 \in S$: $2x+3 = 1(2x+3) + 0(x-1) + 0(-x-4)$

$x-1 \in S$: $x-1 = 0(2x+3) + 1(x-1) + 0(-x-4)$

(b). $-x+6 \stackrel{?}{=} \alpha_1(2x+3) + \alpha_2(x-1) + \alpha_3(-x-4)$. <u>Yes</u>.

Equate like terms: $-1 = 2\alpha_1 + \alpha_2 - \alpha_3$

$6 = 3\alpha_1 - \alpha_2 - 4\alpha_3$

Let $\alpha_1 = 0$. then $-1 = \alpha_2 - \alpha_3$. $\qquad \alpha_2 = -1-1 = -2$.

$6 = -\alpha_2 - 4\alpha_3$.

$5 = -5\alpha_3 \rightarrow \alpha_3 = -1$.

$-x+6 = 0(2x+3) - 2(x-1) - 1(-x-4) \in S$

(c). Prove if $u, v \in V$ then $\text{Span}(u,v)$ is a subspace of V.

(i) $\text{Span}(u,v) \subseteq V$:

Let $x \in S$: $x = \alpha_1 u + \alpha_2 v$ by def $\text{Span}(u,v)$.

$\alpha_1 u \in V$ & $\alpha_2 v \in V$ bec. $u, v \in V$ & V is closed under scalar mult bec. V is a v.s.

$\alpha_1 u + \alpha_2 v \in V$ because $\alpha_1 u, \alpha_2 v \in V$ & V is closed under add$^{\underline{n}}$ bec. V is a v.s

So $\text{Span}(u,v) \subseteq V$.

(ii) Let $x, y \in \text{Span}(u,v)$: $x = \alpha_1 u + \alpha_2 v$, $y = \beta_1 u + \beta_2 v$.

Let $c \in \mathbb{R}$.

(A). $x+y \stackrel{?}{\in} \text{Span}(u,v) \Leftrightarrow$ Do constants c_1 & c_2 exist so that $x+y = c_1 u + c_2 v$?

(5c) cont. $x+y = \alpha_1 u + \alpha_2 v + \beta_1 u + \beta_2 v$

$= (\alpha_1+\beta_1)u + (\alpha_2+\beta_2)v.$

So pick $c_1 = \alpha_1+\beta_1$, $c_2 = \alpha_2+\beta_2$ ✓

$x+y \in \text{Span}(u,v)$, so Span(u,v) is closed under add

(B) $\alpha x \overset{?}{\in} \text{Span}(u,v) \Leftrightarrow$ Do constants c_1 & c_2 exist so that $\alpha x = c_1 u + c_2 v$?

$\alpha x = \alpha\alpha_1 u + \alpha\alpha_2 v$

So pick $c_1 = \alpha \cdot \alpha_1$, $c_2 = \alpha\alpha_2$.

$\alpha x \in \text{Span}(u,v)$, so span(u,v) is closed under scalar mult.

So Span(u,v) is a subspace of V.

(5d). $B = \{x-1, -x-4\}$

Note $B \subseteq S$ ✓

(A) $x-1$ is not a multiple of $-x-4$ so B is linearly indep.

(B) prove B spans S: Let f be an arbitrary element of S: $f = a(2x+3)+b(x-1)+c(-x-4)$.

Show we can pick α_i's so that $f = \alpha_1(x-1)+\alpha_2(-x-4)$.

show $a(2x+3)+b(x-1)+c(-x-4) = \alpha_1(x-1)+\alpha_2(-x-4)$.

Equate like terms: $2a+b-c = \alpha_1 - \alpha_2$. ⊛

$+\ 3a-b-4c = -\alpha_1 - 4\alpha_2.$

$5a-5c = -5\alpha_2.$

Pick $\alpha_2 = c-a$.

⊛ → $\alpha_1 = 2a+b-c+\cancel{\alpha_2}\ c-a = a+b.$

(5d) cont.

We can write an arbitrary elt of S as.

$a(2x+3) + b(x-1) + c(-x-4) = (a+b)(x-1) + (c-a)(-x-4)$

So $\mathcal{B}$ spans S.

Therefore, $\mathcal{B}$ is a basis.

(6). (a) $\mathcal{B} = \{(3,0,-1), (0,1,1)\}$.

(b) $\dim(\mathcal{B}) = 2$.

(7) (a) $B = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$.

(b) $\dim(\mathcal{B}) = 3$

(8). $y_1 = e^{2x} \rightarrow y_1' = 2e^{2x} \rightarrow y_1'' = 4e^{2x}$.

$4e^{2x} + 2e^{2x} - 6e^{2x} = 0$ ✓

$y_2 = e^{-3x} \rightarrow y_2' = -3e^{-3x} \rightarrow y_2'' = 9e^{-3x}$.

$9e^{-3x} - 3e^{-3x} - 6e^{-3x} = 0$ ✓.

(b) Every solution can be written as a linear combination of y_1 & y_2.

(c) Saying every linear combination of y_1 & y_2 is also a sol'n is not the same as saying <u>every</u> solution can be written as a linear combination of y_1 & y_2.