HW 3 – A Sol$^{\underline{n}}$

AGM I
F2007
Gross

(1) $f(x+iy) = \dfrac{x+iy+i}{(x+iy)^2+1} = \dfrac{x+i(1+y)}{x^2-y^2+1+i2xy} \cdot \dfrac{(x^2-y^2+1-2ixy)}{(x^2-y^2+1-i2xy)}$

$$= \frac{x^3 - xy^2 + x + (1+y)2xy}{(x^2-y^2+1)^2 + 4x^2y^2} + i\,\frac{(1+y)(x^2-y^2+1) - 2x^2y}{(x^2-y^2+1)^2 + 4x^2y^2}$$

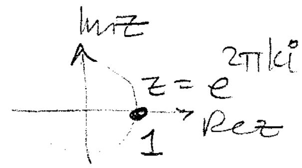
(2)(a) $u(x,y) = \dfrac{x-1}{(x-1)^2+y^2}$, $v(x,y) = \dfrac{2y}{(x-1)^2+y^2}$.

(b). $x \stackrel{?}{=} \dfrac{x+\cancel{iy}+x-\cancel{iy}}{2}$ ✓. $\quad y \stackrel{?}{=} \dfrac{\cancel{x}+iy-(\cancel{x}-iy)}{2i}$ ✓.

(c). $f(z,\bar{z}) = \dfrac{\dfrac{z+\bar{z}}{2} - 1 + \dfrac{\cancel{2i}(z-\bar{z})}{\cancel{2i}}}{\dfrac{(z+\bar{z}-2)^2}{4} + \dfrac{(z-\bar{z})^2}{-4}} \quad \dfrac{4}{4}$

$$= \frac{2(z+\bar{z}) - 4 + z - \bar{z}}{(z+\bar{z}-2)^2 - (z-\bar{z})^2}$$

(3). $e^{z_1} = e^{z_2 + k2\pi i} = e^{z_2}\, \underbrace{e^{2\pi k i}}_{=1}$



$= e^{z_2}$ ✓.

(4). $f(z+p) = e^{z+p} = e^{z+2\pi i} = e^z\, \underbrace{e^{2\pi i}}_{=1} = e^z$

$= f(z)$ ✓.

(5) Find a $z \in \mathbb{C}$ such that $|\sin z| > 1$. (contrast w/ real-valued sine fnc $f(x) = \sin x$!)

$$z = iy$$

$$|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| \quad \text{def of cpx sine}$$

$$= \left| \frac{e^{i(iy)} - e^{-i(iy)}}{2i} \right| \quad \text{def of } z.$$

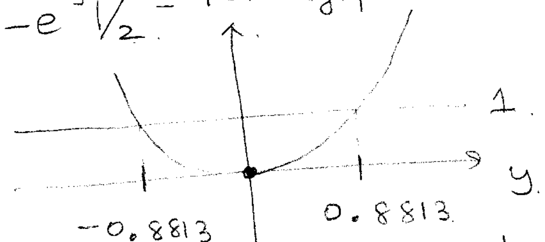
$$= \left| \frac{e^{-y} - e^{y}}{2i} \right|$$

$$= \frac{|e^{-y} - e^{y}|}{|2i|} \quad \text{because } |z_1/z_2| = |z_1|/|z_2|$$

$$= \frac{|e^{-y} - e^{y}|}{2} \leftarrow \text{real}$$

> 1 for y sufficiently large in absolute value, e.g. $y = 2$,

$|e^{-y} - e^{y}|/2 = |\sinh(y)|$



$$\boxed{z = 2i}$$

$|\sin 2i| = |e^{-2} - e^{2}|/2 \approx 3.6 > 1.$

Note $|\sin z| = |e^{-y} - e^{y}|/2$ can be made arbitrarily large; $|\sin(z)|$ is unbounded.

(1i). (a). $z=0$ is removeable:

$\lim_{z\to 0} \frac{1-\cos z}{z}$ has indeterminate form $\frac{0}{0}$.

$\left(\cos 0 = \frac{e^{i\cdot 0}+e^{-i\cdot 0}}{2} = 1\right)$.

$= \lim_{z\to 0} \frac{\sin z}{1}$ (L'Hospital's Rule works for cpx limits)

$= 0$ $\left(\sin 0 = \frac{e^{i0}-e^{-i0}}{2i} = 0\right)$.

limit exists!

(11b) $z=1+i$ is a pole of order 2; $2i$ is a pole of order 3. (11c) $z = 2-3i$ is an essential singularity

(1c). $\cos(z_1+z_2) \stackrel{?}{=} \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

$$\Longleftrightarrow \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2} \stackrel{?}{=} \frac{e^{iz_1}+e^{-iz_1}}{2} \cdot \frac{e^{iz_2}+e^{-iz_2}}{2} + \frac{(e^{iz_1}-e^{-iz_1})}{2\cancel{i}} \cdot \frac{(e^{iz_2}-e^{-iz_2})}{2\cancel{i}}$$

$$= \frac{1}{4}\Big[e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} + \cancel{e^{i(z_2-z_1)}} + \cancel{e^{i(z_1-z_2)}} + e^{i(z_1+z_2)} + e^{-i(z_1+z_2)} - \cancel{e^{i(z_2-z_1)}} - \cancel{e^{i(z_1-z_2)}}$$

$$= \frac{1}{\cancel{4}\,2}\Big[\cancel{2}e^{i(z_1+z_2)} + \cancel{2}e^{-i(z_1+z_2)}\Big].$$

✓

(7) $\overline{(e^{z})} \stackrel{?}{=} e^{\bar{z}}$

$\overline{(e^{z})} = \overline{(e^{x+iy})} = \overline{(e^{x} e^{iy})} =$

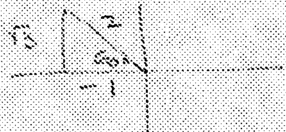
$= \overline{[e^{x}(\cos y + i\sin y)]} = e^{x}(\cos y - i\sin y)$

$= e^{x}(\cos(-y) + i\sin(-y))$

$= e^{x} e^{-iy} = e^{x-iy} = e^{\bar{z}}$ ✓

(8) $\sin(2i) = [e^{i(2i)} + e^{-i(2i)}]/2i$

$= (e^{-2} + e^{2})/2i = -i\cosh(2)$



(9) $\log(-1+i\sqrt{3})$

$= \log(2e^{i2\pi/3 + k2\pi})$

$= \ln 2 + i\left(\frac{2\pi}{3} + k2\pi\right)$, $k \in \mathbb{Z}$

(a) Principal value of $\log z$ is $\ln 2 + i\frac{2\pi}{3}$.

(b) For $k \neq 0$ above, we get other numbers for $\log(-1+i\sqrt{3})$.

(10) $\lim_{\omega\to\infty} \rho[\cos\omega - i\sin\omega]$ does not exist.