Hw 15 — A sol^{n} AEM I

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F2007

(1). Stokes: $\iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \int_C \underline{F} \cdot d\underline{r}$

$$\text{curl}\, \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \underline{i}(0) - \underline{j}(0) + \underline{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$\iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \iint_S \text{curl}\, \underline{F} \cdot \underline{n}\, dS$. Here $\underline{n} = \underline{k}$.

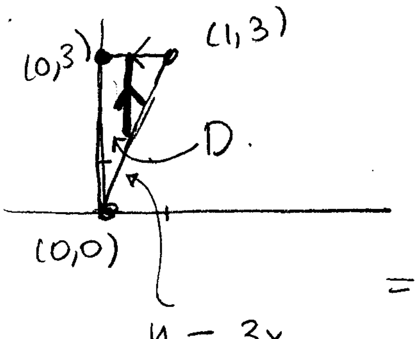
$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$. The projection D of the region S into the xy-plane is D=S itself.

$$\int_C \underline{F} \cdot d\underline{r} = \int_C (P\underline{i} + Q\underline{j} + 0\underline{k}) \cdot (dx\, \underline{i} + dy\, \underline{j} + dz\, \underline{k})$$

$$= \int_C P\,dx + Q\,dy$$

So $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$ by Stokes.

(2).



$$\int \underbrace{x^2y^2}_{P}\,dx + \underbrace{4xy^3}_{Q}\,dy =$$

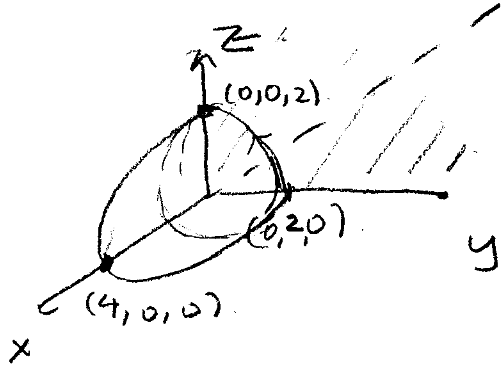
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \text{ by Green}$$

$$= \int_0^1 \int_{3x}^{3} (4y^3 - 2x^2y)\, dy\, dx =$$

$$= \int_0^1 (y^4 - x^2y^2)\Big|_{y=3x}^{3} dx = \int_0^1 \left(3^4 - 9x^2 - \overset{72}{\cancel{3^4}}x^4 \cancel{+9x^4}\right) dx$$

$$= \left(3^4x - x^3 3 - \frac{72x^5}{5}\right)\Big|_0^1 = \frac{3^4 \cdot 5 - 15 - 72}{5} = \frac{318}{5}$$

(3).



$$x = 4 - y^2 - z^2$$

$$\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\ dA$$

(a). $\iint_{\cancel{S}\ D} (y^2 + z^2)\ \cancel{dS}$

$$= \int_0^{2\pi}\int_0^{2} \underbrace{(\cancel{y^2+z^2})}_{r^2} \sqrt{\underbrace{\cancel{4y^2+4z^2}}_{4r^2}+1}\ r\,dr\,d\theta$$

$y = r\cos\theta$

$z = r\sin\theta$

$u = 4r^2 + 1 \longrightarrow r^2 = \frac{u-1}{4}$

$du = 8r\,dr$

$$= \int_0^{2\pi} d\theta \int_1^{17} \frac{1}{8}\frac{1}{4}(u-1)\,u^{1/2}\,du$$

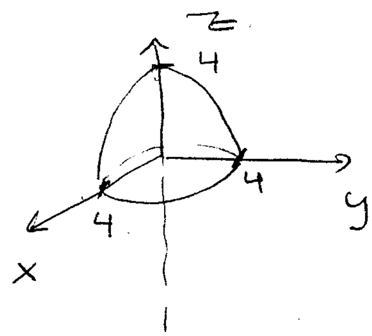
$$= \frac{2\pi}{32}\int_1^{17}\left(u^{3/2} - u^{1/2}\right)du = \frac{\pi}{16}\left[u^{5/2}\,\frac{2}{5} - u^{3/2}\,\frac{2}{3}\right]_1^{17}$$

$$= \frac{\pi}{16}\left[\frac{2}{5}\,17^{5/2} - \frac{2}{3}\,17^{3/2} - \frac{2}{5} + \frac{2}{3}\right] \quad g$$

(b). $\int_0^{2\pi}\int_0^{2} \sqrt{4r^2+1}\ r\,dr\,d\theta$ (set $\rho \equiv 1$).

$$= 2\pi\ \frac{(4r^2+1)^{3/2}}{8}\,\frac{2}{3}\Bigg|_0^2 = \frac{\pi}{6}\left[17^{3/2} - 1\right]\ cm^2$$

(4). $\iint_S \rho \underline{v}\cdot d\underline{S} = \iint_S \rho\underline{v}\cdot\underline{n}\,dS$



$\rho\underline{v} = -y\,\underline{i} + x\,\underline{j} + 3z\,\underline{k}$

$z = g(x,y) = \sqrt{16-x^2-y^2}$

$w = z - g(x,y)$

$$\underline{n} = \frac{\underline{\nabla} w}{|\underline{\nabla} w|} = \frac{-\frac{\partial g}{\partial x}\underline{i} - \frac{\partial g}{\partial y}\underline{j} + \underline{k}}{\sqrt{g_x^2+g_y^2+1}}$$

$$\iint_D \left(+y\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial y} + 3z\right) dA$$

$$\frac{\partial g}{\partial x} = \frac{1}{\cancel{2}}\,\frac{-2x}{\sqrt{16-x^2-y^2}}, \quad \frac{\partial g}{\partial y} = \frac{-y}{\sqrt{16-x^2-y^2}}$$

$$\int_0^{2\pi}\int_0^4 \left(\cancel{\frac{-2xy}{\sqrt{16-x^2-y^2}}} + \cancel{\frac{2xy}{\sqrt{16-x^2-y^2}}} + 3\sqrt{16-x^2-y^2}\right) r\,dr\,d\theta$$

$$= 2\pi\cdot\cancel{3}\cdot\frac{2}{\cancel{3}}(16-r^2)^{3/2}\left(\frac{-1}{\cancel{2}}\right)\Big|_0^4$$

$= -2\pi[0-4^3] = 128\pi$ units mass/time

(5) $\varepsilon_0 \iint_S \underline{E}\cdot d\underline{S} = \varepsilon_0\iiint_E \underline{\nabla}\cdot\underline{E}\,dV = \varepsilon_0\iiint_E 3\,dV$

$= 3\varepsilon_0\,\mathrm{vol}(E) = 3\varepsilon_0\, l\cdot w\cdot h = 3\varepsilon_0\, 2^3 = 24\,\varepsilon_0$

units charge.

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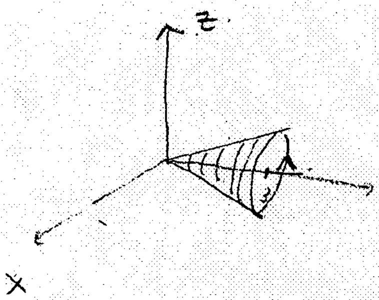
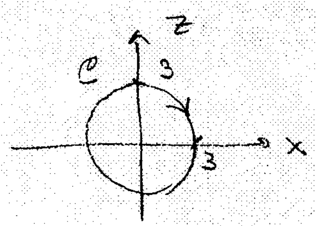
(6) Use Stokes' Thm to evaluate $I = \iint_S (\nabla \times \underline{F}) \cdot d\underline{S}$ if $\underline{F} = x^2 y^3 z\, \underline{i} + \sin(xyz)\, \underline{j} + xyz\, \underline{k}$, & S is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ & $y = 3$ oriented in the direction of the positive y-axis.



(2) $$I = \int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \underline{F}(t) \cdot \underline{r}'(t)\, dt$$

$$= \int_0^{2\pi} \left(3^6 \sin^2 t \cos t\, \underline{i} + \sin(3^3 \sin t \cos t)\, \underline{j} + 3^3 \sin t \cos t\, \underline{k}\right) \cdot (+3\cos t\, \underline{i} - 3 \sin t\, \underline{k})\, dt$$



$$= \int_0^{2\pi} \left(+3^7 \sin^2 t \cos^2 t \overset{(*)}{} - 3^4 \sin^2 t \cos t\right) dt$$

$9 = x^2 + z^2$

$$= +3^7\left(\frac{1}{8}t - \frac{\sin 4t}{32}\right) - 3^4 \frac{\sin^3 t}{3}\Bigg|_0^{2\pi}$$

(1) $x = 3 \sin t$
$z = 3 \cos t$
$y = 3$

$$= \boxed{+3^7\left(\frac{\pi}{4}\right)}$$

$\underline{r}(t) = 3 \sin t\, \underline{i} + 3\, \underline{j} + 3 \cos t\, \underline{k}$

(*)
$$[\sin t \cos t]^2 = \left[\tfrac{1}{2} \sin(2t)\right]^2$$
$$= \frac{1}{4} \cdot \frac{1 - \cos 4t}{2}$$
$$= \frac{1}{8}(1 - \cos 4t)$$

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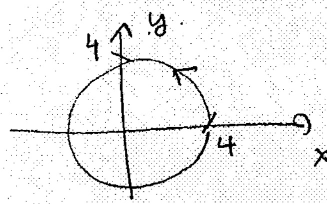
(7) $\underline{F} = yz\,\underline{i} + 2xz\,\underline{j} + e^{xy}\,\underline{k}$ & C is the circle $x^2+y^2=16$, $z=5$



$z = 5$

(A) $\iint_S \nabla\times\underline{F} \cdot dS$

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & e^{xy} \end{vmatrix} =$$

$$\underline{i}\,(e^{xy}x - 2x) - \underline{j}\,(e^{xy}y - y) + \underline{k}\,(2z - z)$$

$$\underline{n} = \underline{k} \;;\; z = g(x,y) \equiv 5 \rightarrow \sqrt{\left(\tfrac{\partial g}{\partial x}\right)^2 + \left(\tfrac{\partial g}{\partial y}\right)^2 + 1} = 1$$

$$\iint_S \nabla\times\underline{F}\cdot\underline{n}\,dS = \iint_S z\,dS = \iint_D 5\,dA =$$

$$5\cdot\pi\cdot 16 = \boxed{80\pi}$$

(B) $\underline{r}(t) = 4\cos t\,\underline{i} + 4\sin t\,\underline{j} + 5\,\underline{k}$

$\underline{r}'(t) = -4\sin t\,\underline{i} + 4\cos t\,\underline{j} + 0\,\underline{k}$

$$\int_0^{2\pi} \underline{F}\cdot\underline{r}'(t)\,dt = \int_0^{2\pi} \left(20\sin t\,\underline{i} + 40\cos t\,\underline{j} + e^{16\sin t\cos t}\,\underline{k}\right) \cdot \left(-4\sin t\,\underline{i} + 4\cos t\,\underline{j} + 0\,\underline{k}\right) dt$$

$$= \int_0^{2\pi} \left[\sin^2 t\,(-80) + 160\cos^2 t\right] dt$$

$$= 80\int_0^{2\pi} -\left(\frac{1-\cos 2t}{2}\right) + \cancel{2}\left(\frac{1+\cos 2t}{\cancel{2}}\right) dt$$

$$= 80\int_0^{2\pi} \frac{1}{2} + \frac{3\cos 2t}{2}\,dt = 80\left[\frac{1}{2}t + \frac{3}{2}\,\frac{\sin 2t}{2}\right]_0^{2\pi} = \boxed{80\pi}$$

(A) = (B) ✓

(8)

Hw 15.6

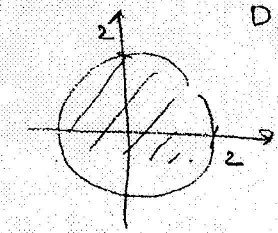
(A)

$$I = \iint_S \underline{F} \cdot d\underline{S} = \iint_S \underline{F} \cdot \underline{n}\, dS$$

$$z = 4 - x^2 - y^2 = g(x,y).$$

$$z - g(x,y) = f(x,y,z) = 0$$

D.



$$0 = 4 - x^2 - y^2$$

$$\underline{n} = \frac{\nabla f}{|\nabla f|} = \frac{-\frac{\partial g}{\partial x}\underline{i} - \frac{\partial g}{\partial y}\underline{j} + \underline{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

$$I = \iint_D \left(x^2\underline{i} + xy\,\underline{j} + (4 - x^2 - y^2)\underline{k}\right) \cdot \frac{(2x\,\underline{i} + 2y\,\underline{j} + \underline{k})}{\cancel{\sqrt{g_x^2 + g_y^2 + 1}}} \cancel{\sqrt{g_x^2 + g_y^2 + 1}}\; dA$$

$$= \iint_D \left[2x^3 + 2xy^2 + (4 - x^2 - y^2)\right] dA$$

$$\cos\theta \cdot \cos^2\theta = \cos\theta(1 - \sin^2\theta)$$

$$= \int_0^{2\pi}\int_0^{2} \left[2r^3 \cos^3\theta + 2r^3\cos\theta\sin^2\theta + (4 - r^2)\right] r\,dr\,d\theta.$$

$$= \int_0^{2} \left[2r^3 \sin\theta + (4 - r^2)\,\theta\right]_{\theta=0}^{2\pi} r\,dr$$

$$= \int_0^{2} (4r - r^3)\, 2\pi\, dr = \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 \cdot 2\pi.$$

$$= 2\pi\,(8 - 4) = \boxed{8\pi} \checkmark$$

(B).

$$\iiint_V \nabla \circ \underline{F} \, dV$$

$$\nabla \circ \underline{F} = 2x + x + 1 = 3x + 1$$

$$= \iint_D \left[\int_0^{4-x^2-y^2} (3x+1) \, dz \right] dA$$

$$= \iint_D (3x+1)(4-x^2-y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^{2} (3r\cos\theta + 1)(4-r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2} (3r\sin\theta + \theta)(4-r^2) r \Big|_{\theta=0}^{2\pi} dr$$

$$= \int_0^{2} 2\pi (4-r^2) r \, dr = 2\pi \int_0^2 (4r - r^3) \, dr =$$

$$= 2\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2 = 2\pi (8-4) = \boxed{8\pi} \checkmark$$

(A) = (B) ✓

(9) $\iint_S u\nabla v \cdot d\underline{S} = \iiint_E \underline{\nabla}\cdot(u\underline{\nabla} v)\, dV$ by

Divergence thm.

& $\underline{\nabla}\cdot(u\underline{\nabla} v) = \underline{\nabla} u \cdot \underline{\nabla} \underline{v} + u\Delta \underline{v}$ (see below)**,

so Green's First Identity holds.

** $\underline{\nabla}\cdot(u\underline{\nabla} v) = \underline{\nabla}\cdot\left(u \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \right\rangle\right)$

$= \underline{\nabla}\cdot\left\langle u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y}, u\frac{\partial v}{\partial z}\right\rangle.$

$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} +$

$+ \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + u\frac{\partial^2 v}{\partial z^2}. \longleftarrow (*).$

$\underline{\nabla} u \cdot \nabla \underline{v} + u\Delta \underline{v} = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right\rangle \cdot \left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\right\rangle$

$+ u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$

$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$

$+ u\frac{\partial^2 v}{\partial x^2} + u\frac{\partial^2 v}{\partial y^2} + u\frac{\partial^2 v}{\partial z^2}. = (*)$ ✓