HW 12 – A sol$^{\underline{n}}$

AEM I
Gross.

(1) $my'' + \gamma y' + ky = 0$; $u = y$, $v = y'$

(a) $\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$ $\Leftarrow \begin{cases} u' = v \\ mv' + \gamma v + ku = 0 \end{cases}$

(b) $\begin{vmatrix} -\lambda & 1 \\ -k/m & -\gamma/m - \lambda \end{vmatrix} = \lambda^2 + \frac{\gamma}{m}\lambda + \frac{k}{m} = 0.$

$\Leftrightarrow$ $m\lambda^2 + \gamma\lambda + k = 0 \rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$

spiral point if $\underline{\gamma^2 < 4mk}$. ← underdamped

$\underline{\text{asymptotically stable}}$ because $Re(\lambda) = -\gamma/2m < 0$.

(c) $\gamma^2 = 4mk$ ← critically damped.

$\lambda_1 = \lambda_2 = -\frac{\gamma}{2m} < 0 \rightarrow$ $\underline{\text{asymptotically stable}}$

$\begin{bmatrix} \frac{\gamma}{2m} & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} + \frac{\gamma}{2m} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 2m \\ -\gamma \end{bmatrix}$. check:

$\frac{\gamma}{2m} 2m - \gamma(1) = 0$ ✓

$-\frac{k}{m} \cdot 2m - \gamma\left(-\frac{\gamma}{m} + \frac{\gamma}{2m}\right) = \frac{-k\,4m}{2m} + \underbrace{\cancel{\gamma}^{2}}_{4mk}\left(\frac{+2-1}{2m}\right)$

$= \dfrac{-4km + 4mk}{2m} = 0$ ✓.

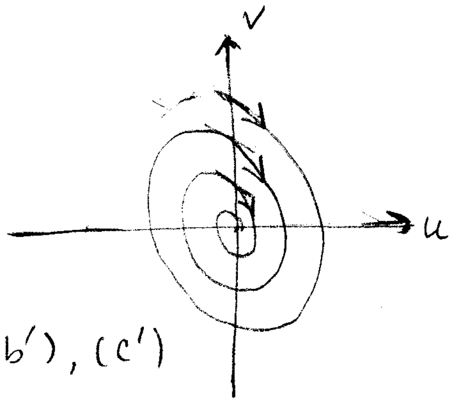
only one linearly indep eigenvector → $\underline{\text{proper node}}$

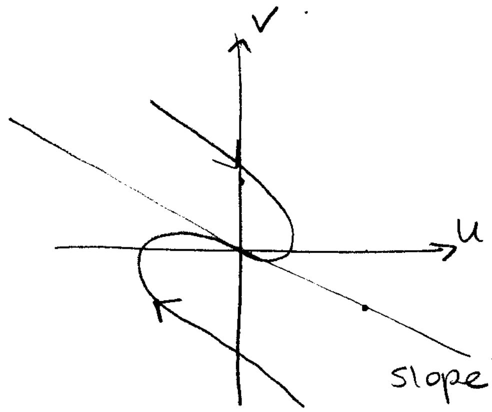
(d). $\gamma^2 > 4mk$ ← overdamped.

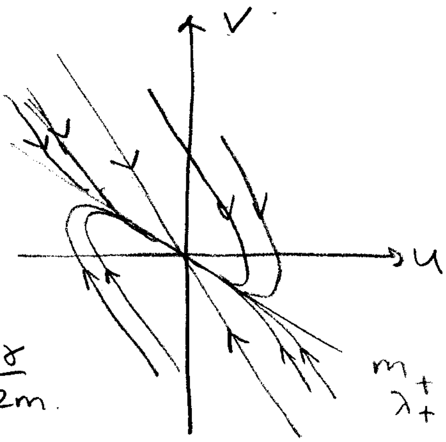
$\left.\begin{array}{l} \lambda_- = \dfrac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < 0. \\ \lambda_+ = \dfrac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < 0 \end{array}\right\} \Rightarrow$ $\underline{\text{asymptotically Stable}}$ $\underline{\text{improper node}}$.

$(\gamma^2 - 4mk) < \gamma^2$.

$(\sqrt{D} < \gamma)$.







(b′), (c′)

$$\underline{u}'\Big|_{(0,1)} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\gamma/m \end{bmatrix}$$

$du/dt > 0$

$$\frac{dv}{du} = \frac{-\gamma/m}{1}$$

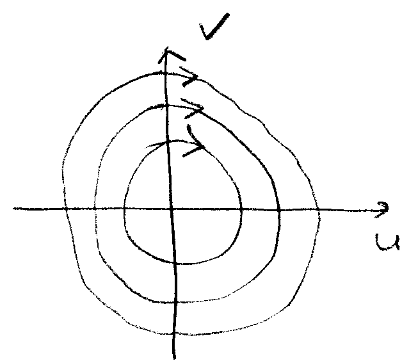
(d′) $\lambda = \lambda_\pm$: $\begin{bmatrix} \frac{\gamma}{2m} \mp \frac{\sqrt{D}}{2m} & 1 \\ -\frac{k}{m} & -\frac{\gamma^2}{m^2} + \frac{\gamma}{2m} \mp \frac{\sqrt{D}}{2m} \end{bmatrix} \underline{X} = \underline{0}$

if $\underline{X} = \begin{bmatrix} 1 \\ -\frac{\gamma}{2m} \pm \frac{\sqrt{D}}{2m} \end{bmatrix}$ $\quad D = \gamma^2 - 4mk$; $0 < D < \gamma$

$m_\pm$

(f) $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}\Bigg|_{\gamma = 0}$

$$= \frac{\pm 2\sqrt{mk}\, i}{2m} = \pm\sqrt{\frac{k}{m}}\, i$$



$$\underline{u}'\Big|_{(0,1)} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\frac{du}{dt} > 0$

$dv/du = \frac{0}{1}$

The mass oscillates with constant amplitude for all time.

(2). (a). $\underline{x}^{(1)} = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + i e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

$\underline{x}^{(2)} = e^{(1-2i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = e^t e^{-2it} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$= e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ (\cos 2t - i \sin 2t)(1+i) \end{bmatrix}$

$= e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - i e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

$= \bar{\underline{x}}^{(1)}$

(b). $\underline{x}^{(3)} = \dfrac{\underline{x}^{(1)} + \underline{x}^{(2)}}{2} = Re(\underline{x}^{(1)}) = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$

$x^{(4)} = \dfrac{\underline{x}^{(1)} - \underline{x}^{(2)}}{2i} = \frac{1}{\cancel{2i}} \cancel{i2} Im(\underline{x}^{(1)}) = e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$

$\underline{x} = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$

$+ c_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}.$

$F = -x^2 + \frac{3}{2}x - \frac{1}{2}xy \qquad G = -y^2 + 2y - \frac{3}{4}xy$

(3) $x' = x\left(\frac{3}{2} - x - \frac{1}{2}y\right), \quad y' = y\left(2 - y - \frac{3}{4}x\right)$

(a) $x\left(\frac{3}{2} - x - \frac{1}{2}y\right) = 0$

$x = 0$ or $\frac{3}{2} - x - \frac{1}{2}y = 0$

$x = 0$: $y = 0$ or $2 - y = 0$

$(0,0)$ Ⓐ, $(0,2)$ Ⓑ

$\frac{3}{2} - x - \frac{1}{2}y = 0$: $y = 0$ or $+x + \frac{1}{2}y = +\frac{3}{2}$

$\frac{3}{2} - x = 0$ Ⓒ $\left(\frac{3}{2}, 0\right)$.

$-\frac{4}{3}\left[\frac{3}{4}x + y = 2\right]$ Ⓓ $\left(\frac{4}{5}, \frac{7}{5}\right)$

(b) $\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} -2x + \frac{3}{2} - \frac{1}{2}y & -\frac{1}{2}x \\ -\frac{3}{4}y & -2y + 2 - \frac{3}{4}x \end{bmatrix} = B$

Ⓐ $B\Big|_{\boxed{(0,0)}} = \begin{bmatrix} 3/2 & 0 \\ 0 & 2 \end{bmatrix}$

$\underline{\underline{\lambda = \frac{3}{2}}}$: $\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{\underline{\underline{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}}$

$\underline{\underline{\lambda = 2}}$: $\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{\underline{\underline{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}}$

Ⓑ $B\Big|_{\boxed{(0,2)}} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & -2 \end{bmatrix}$

$\underline{\underline{\lambda = \frac{1}{2}}}$: $\begin{bmatrix} 0 & 0 \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{\underline{\underline{X} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}}}$

$\underline{\underline{\lambda = -2}}$: $\begin{bmatrix} \frac{5}{2} & 0 \\ -\frac{3}{2} & 0 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{\underline{\underline{X} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}}$

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(C) $B\big|_{\left(\frac{3}{2},0\right)} = \begin{bmatrix} -\frac{3}{2} & -\frac{3}{4} \\ 0 & 7/8 \end{bmatrix}$

$\underline{\lambda = -\frac{3}{2}}$: $\begin{bmatrix} 0 & -\frac{3}{4} \\ 0 & \frac{19}{8} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$204 = 2 \cdot 102$, $102 = 2 \cdot 51$

$\underline{\lambda = \frac{7}{8}}$: $\begin{bmatrix} -\frac{19}{8} & -\frac{3}{4} \\ 0 & 0 \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} -6 \\ 19 \end{bmatrix}$

(D) $B\big|_{\left(\frac{4}{5},\frac{7}{5}\right)} = \begin{bmatrix} -\frac{4}{5} & -\frac{2}{5} \\ -\frac{21}{20} & -\frac{7}{5} \end{bmatrix}$ $\quad \left(\frac{4}{5}+\lambda\right)\left(\frac{7}{5}+\lambda\right) - \frac{21(\cancel{2})}{\cancel{100}_{50}}$

$= \frac{28}{25} - \frac{21}{50} + \lambda^2 + \frac{11}{5}\lambda = \lambda^2 + \frac{11}{5}\lambda + \frac{\cancel{35}^{7}}{\cancel{50}_{10}} = 0.$

$280 - 21 = 259$ $\quad 56 - 21 = 35$

$10\lambda^2 + 22\lambda + 7 = 0 \to \lambda = \frac{-22 \pm \sqrt{204}}{20} = \frac{-11 \pm \sqrt{51}}{10}$

$m \approx -1.04$

$\underline{\lambda = \frac{-11+\sqrt{51}}{10}}$: $\begin{bmatrix} -\frac{8}{10} + \frac{11-\sqrt{51}}{10} & -\frac{2}{5}\cdot\frac{2}{2} \\ -\frac{21}{20} & -\frac{7}{5} + \frac{11}{10} - \frac{\sqrt{51}}{10} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 4 \\ 3-\sqrt{51} \end{bmatrix}$.

$\underline{\lambda = \frac{-11-\sqrt{51}}{10}}$: $\begin{bmatrix} \frac{-8+11+\sqrt{51}}{10} & -\frac{4}{10} \\ -\frac{21}{20} & -\frac{7}{5} + \frac{11}{10} + \frac{\sqrt{51}}{10} \end{bmatrix} \underline{X} = \underline{0}$ if $\underline{X} = \begin{bmatrix} 4 \\ 3+\sqrt{51} \end{bmatrix}$.

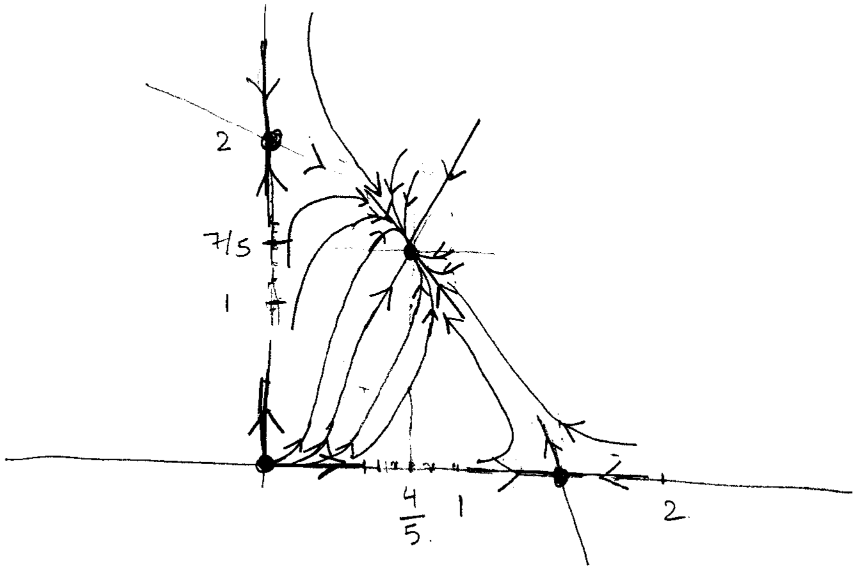
$m \approx 2.55$

(c). (0,0) : unstable improper node

(0, 2) : unstable saddle

($\frac{3}{2}$, 0) : unstable saddle

(4/5, 7/5): asymptotically stable improper node (d).



(e) $\lim_{t\to\infty} x = \frac{4}{5}$, $\lim_{t\to\infty} y = \frac{7}{5}$ if we start with nonzero amt of each species.

(f) basin of attraction for $(4/5, 7/5)$ is whole interior of first quadrant.

(g). There are not multiple basins of attraction (because there are not multiple asymptotically stable critical points), so there is no separatrix to separate them.

(h). In the long run, the species will level out at constant nonzero levels (4/5 for x & 7/5 for y), assuming we start with a nonzero amount of each species. They peacefully coexist in the environment in which they compete for resources; neither becomes extinct.