

1. **Elementary equation operations**, which produce an *equivalent system*, include
  - (a) **Interchange**: Interchange two equations:  $E_i \leftrightarrow E_j$ ,
  - (b) **Scaling**: Multiply (or divide) an equation by a nonzero constant to produce a new Equation  $i$ :  $cE_i \rightarrow E_i$ ,
  - (c) **Elimination**: Add a constant multiple of another equation to Equation  $i$  to produce a new Equation  $i$ :  $E_i + cE_j \rightarrow E_i$ .
2. **Elementary row operations** come from replacing “equation” with “row” and “E” with “R” in the definition above to produce the following:
  - (a) **Interchange**: Interchange two rows:  $R_i \leftrightarrow R_j$ ,
  - (b) **Scaling**: Multiply (or divide) a row by a nonzero constant to produce a new Row  $i$ :  $cR_i \rightarrow R_i$ ,
  - (c) **Elimination**: Add a constant multiple of another row to Row  $i$  to produce a new Row  $i$ :  $R_i + cR_j \rightarrow R_i$ .
3. The **leading entry** in a row means the *first nonzero entry* in the row.
4. A matrix is in reduced row echelon form (**RREF**) if and only if it has the properties below:
  - (a) **Leading-1 property**: The leading entry in every nonzero row is a 1,
  - (b) **Unit-column property**: Every column containing a leading entry is a *unit column*, meaning a column containing only zeros except for one 1,
  - (c) **Zero-row property**: All zero rows (rows containing only zeros) are on the bottom,
  - (d) **Stairstep property**: The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.
5. **Gaussian elimination** (or Gauss elimination) is an algorithm (procedure) consisting of row operations that produce a matrix in RREF.
  - (a) **Step 1: Find the leftmost nonzero column.**
  - (b) **Step 2: In the top row, get a nonzero entry (called a pivot) in the column of Step 1.** (Either a nonzero entry will already be there, or you will have to interchange a row with the top row.)
  - (c) **Step 3: Obtain zeros below the leading entry of the top row** using elimination. (Specifically, do the elimination  $R_j + cR_t \rightarrow R_j$  for all rows  $R_j$  below the top row  $R_t$ .)
  - (d) **Step 4: Cover the top row.** If the remaining submatrix contains more than one row, **repeat Steps 1–3 on the remaining submatrix.** Otherwise go to Step 5.
  - (e) **Step 5: In the last nonzero row, use scaling to obtain a leading 1.**
  - (f) **Step 6: Using elimination, introduce zeros above the leading 1** from Step 5—unless, of course, the leading 1 is in the top row of the original matrix. (In that case, stop: The matrix is in RREF. Otherwise go on to Step 7.)
  - (g) **Step 7: Cover the bottom row. Repeat Steps 5–6 on the remaining submatrix.**

Pivot-related definitions:

(Note that because the matrix resulting from Gaussian elimination is in RREF, the pivots form a stairstep pattern.)

- A column that contains a pivot is a **pivot column**. (A row that contains a pivot is a **pivot row**.)
- Each column in a coefficient matrix corresponds to a variable in the linear system. A column that contains *no pivot* corresponds to a **free variable**. (A column that does contain a pivot corresponds to a **leading variable**.)
- The **rank** of a matrix  $A$  is the number of pivots that its corresponding RREF matrix contains.

6. A method for solving a linear system involves

- (a) **Step 1: Apply Steps 1–4 of Gaussian elimination** (a forward pass) to the augmented matrix of the system. If at any stage of the process a row becomes

$$[0 \ 0 \ \cdots \ 0 \mid c], \quad c \neq 0,$$

stop. The system is inconsistent.

- (b) **Step 2: Complete the Gaussian elimination** with Steps 5–7 (the backward pass). Write the linear system that corresponds to the matrix in RREF (ignoring any zero rows).
- (c) **Step 3: Write down the solution** by first identifying the free variables (if any) and assigning them parameter values (say  $r, s, t$ , etc.). Then write the values of the leading variables (in terms of the free variables, if applicable).

7. Note that a homogeneous linear system *always* has the **zero solution** or **trivial solution** (all variables equal to zero) since each equation has the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0.$$

8. There are three possibilities for the number of solutions to linear systems.

- (a) **There is no solution.** (The system in echelon form contains a contradictory equation corresponding to a row of the form  $[0 \ 0 \ \cdots \ 0 \mid c]$ ,  $c \neq 0$ .) **Note a homogeneous system will never have a row of this form.**
- (b) **The system has infinitely many solutions.** (RREF shows there are free variables and no contradictory rows.)
- (c) **The system has a unique solution.** (RREF shows there are *no* free variables and no contradictory rows.)