3450:439/539:001 **Homework 8** Spring 2008

Due: Thursday, March 13, 2007

1. In this problem you will explore simplifications in the Fourier series for $f(x)$ if $f(x)$ is an even or odd function.

   (a) Suppose until further notice that $f(x)$ is an odd function. This means $f(-x) = -f(x)$. The graph is anti-symmetric about the y-axis, meaning that if you reflect the graph for positive x values across the y- and then the x-axis, you get the graph of $f(x)$ for negative x values. Note that $\cos(n\pi x/L)$ is an even function, meaning $\cos(-n\pi x/L) = \cos(n\pi x/L)$. The graph is symmetric about the y-axis. The product $f(x)\cos(n\pi x/L)$ is odd. Explain why the product of any odd function $f(x)$ with any even function $g(x)$ is odd by showing that $fg(-x) = -fg(x)$.

   (b) Show why the integral of an odd function from $-L$ to L is zero. Note this implies the Fourier coefficients

   $$a_n = 0, \quad n = 0, 1, 2, \ldots, \quad \text{for f(x) odd.}$$

   (c) Note that $\sin(n\pi x/L)$ is an odd function. The product $f(x)\sin(n\pi x/L)$ is even. Explain why the product of any odd function $f(x)$ with any odd function $g(x)$ is even by showing that $fg(-x) = fg(x)$.

   (d) Show why the integral of an even function from $-L$ to L is double the value of the integral from 0 to L. Note this implies the Fourier coefficients

   $$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots, \quad \text{for f(x) odd.}$$

   The Fourier series for $f(x)$ odd is called the **Fourier sine series**. Note we can also construct the Fourier sine series for $f(x)$ defined only on $(0, L)$. It converges to the odd $2L$-periodic extension of $f(x)$ except at points of discontinuity, where it converges to the average values of the left and right limits.

   (e) Consider the function $f(x) = 1 - x$, $0 < x < 1$. Graph the function to which its **Fourier sine series** converges. Include three periods.

   (f) Find the **Fourier sine series** of $f(x)$.

   (g) Where is the convergence uniform?

(OVER)

Optional (not for course credit): Find the fifth partial sum of the Fourier sine series, and graph it using a graphing calculator, Mathcad, etc. Point out any places where Gibbs' phenomenon is apparent.

(h) Now suppose until further notice that $f(x)$ is an even function. Recall that $\cos(n\pi x/L)$ is an even function also. Explain why the product of any two even functions $f(x)$ and $g(x)$ is even by showing that $fg(-x) = fg(x)$.

(i) Explain why the Fourier coefficients

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, \ldots, \quad \text{for f(x) even.}$$

(j) Explain why the Fourier coefficients

$$b_n = 0, \quad n = 0, 1, 2, \ldots, \quad \text{for f(x) even.}$$

The Fourier series for $f(x)$ even is called the **Fourier cosine series**. Note we can also construct the Fourier cosine series for $f(x)$ defined only on $(0, L)$. It converges to the even $2L$-periodic extension of $f(x)$ except at points of discontinuity, where it converges to the average values of the left and right limits.

(k) Consider the function $f(x)$ from Problem (1e). Graph the function to which its **Fourier cosine series** converges. Include three periods.

(l) Find the **Fourier cosine series** of $f(x)$.

(m) Where is the convergence uniform?

Optional (not for course credit): Find the fifth partial sum of the Fourier cosine series, and graph it using a graphing calculator, Mathcad, etc. Point out any places where Gibbs' phenomenon is apparent.

2. Consider the function

$$f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1. \end{cases} .$$

(a) Graph the function to which its Fourier series converges. Include three periods.

(b) Find the **phase angle form** of the Fourier series.

(c) Plot some points of the amplitude spectrum.