3450:439/539:001 **Homework 6** Spring 2008

Recommended due date: Thursday, February 28, 2008

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

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1. Consider the differential equation $y'' + \lambda y = 0$ on $-L < x < L$, subject to periodic boundary conditions, namely $y(-L) = y(L)$, $y'(-L) = y'(L)$.

   (a) Show that $\lambda = 0$ is an eigenvalue of this regular Sturm-Liouville eigenvalue problem, and find the corresponding eigenfunction $y_0(x)$.

   (b) Show that there are no negative eigenvalues λ.

   (c) Suppose $\lambda > 0$. Then the boundary-value problem (BVP) will have two infinite families of solutions $cy_1(x)$ and $cy_2(x)$ for appropriate choices of λ. Find the positive eigenvalues λ_n. For each one, find two linearly independent eigenfunctions $y_{1n}(x)$ and $y_{2n}(x)$.

   (d) Verify that all the distinct eigenfunctions you indentified in this problem are mutually orthogonal:

   $$\langle y_n(x), y_m(x) \rangle = \int_a^b y_n(x) y_m(x) r(x)\, dx = 0, \quad n \neq m,$$

   for a, b, and $r(x)$ chosen appropriately for the regular Sturm-Liouville problem under consideration.

2. Consider the regular Sturm-Liouville problem $y'' + \lambda y = 0$ on $0 < x < 1$, $\alpha y(0) + y'(0) = 0$, $y(1) = 0$, where α is a given real constant.

   (a) For all values of α, show there is an infinite sequence of positive eigenvalues λ_n. You may show this graphically for eigenvalues defined via an implicit equation.

   (b) For $\alpha < 1$, show that all real eigenvalues are positive. Show that the smallest eigenvalue approaches zero as α approaches 1 from below.

   (c) Show $\alpha = 1$ is the only α value for which $\lambda = 0$ is an eigenvalue.

   (d) For $\alpha > 1$, show that there is exactly one negative eigenvalue and that this eigenvalue decreases as α increases.

(OVER)

3. Find the forms of eigenfunction expansions of a function $f(x)$ (where $f(x)$ is is piecewise smooth on the set of all real numbers) using the eigenfunctions of the Sturm-Liouville problems in Homework 5 (a)–(c). Show the forms of the coefficients using inner products, and define each inner product $\langle g, h\rangle$ in terms of an integral. On what interval is each expansion valid?