

3450:439/539:001 Homework 5 Spring 2008

Due: Thursday, February 21, 2008

1. Consider the differential equation

$$y'' + \gamma y = 0.$$

- (a) If $\gamma = 1$ and the boundary conditions are $y(0) = 0$ and $y(1) = 0$, show that the only solution to the boundary-value problem is the zero solution.
- (b) If $\gamma = \pi^2$ and the boundary conditions are $y(0) = 0$ and $y(1) = 0$, find an infinite family of solutions (of the form $cy(x)$, where c is an arbitrary constant).
- (c) If $\gamma = \pi^2$ and the boundary conditions are $y(0) + y(1) = 0$ and $y'(0) + y'(1) = 0$, find two infinite families of solutions (of the form $cy_1(x)$ and $cy_2(x)$, where c is an arbitrary constant, and $y_1(x)$ and $y_2(x)$ are linearly independent, meaning not multiples of each other).
- (d) For the boundary conditions in each part (a), (b), and (c) above, are the boundary conditions linear or nonlinear? Are they homogeneous or nonhomogeneous? Are they separated, periodic, or neither?

2. Consider the differential equation

$$P(x)y'' + Q(x)y' + R(x; \lambda)y = 0. \quad (1)$$

We seek an integrating factor $\mu(x)$ such that, upon multiplying equation (1) by $\mu(x)$, the resulting equation can be written in the form

$$[\mu(x)P(x)y']' + \mu(x)R(x; \lambda)y = 0. \quad (2)$$

(Notice that equation (2) can be written in Sturm-Liouville form

$$[p(x)y']' + [-q(x) + \lambda r(x)]y = 0 \quad (3)$$

if we chose $p(x) = \mu(x)P(x)$, and if it is possible to choose $q(x)$ and $r(x)$ such that $\mu(x)R(x; \lambda) = -q(x) + \lambda r(x)$. Otherwise we will not be able to find an appropriate $\mu(x)$ to get Sturm-Liouville form.)

Multiply the equation (1) through by the integrating factor $\mu(x)$ (to be determined). Compare the result to the target equation (2) after applying the product rule of differentiation to (2). Show that μ must be a solution of $P(x)\mu' = (Q(x) - P'(x))\mu$.

The equation you found has solution $\mu = \frac{1}{P} \exp(\int \frac{Q}{P} dx)$. With this choice of μ , (1) and (2) are equivalent. **(OVER)**

3. Write Bessel's equation of order ν ($x > 0$) in the form of a Sturm-Liouville differential equation (3). After determining $\mu(x)$, identify $p(x)$, $q(x)$, and $r(x)$ if ν^2 is denoted as $-\lambda$.

Notice p , p' , q , and r are all continuous on $(0, \infty)$ but are not all continuous at $x = 0$. Also, p and r are both strictly positive on $(0, \infty)$ but are not both strictly positive at $x = 0$.

4. Write Legendre's equation of order α ($-1 < x < 1$) in the form of a Sturm-Liouville differential equation (3). After determining $\mu(x)$, identify $p(x)$, $q(x)$, and $r(x)$ if $\alpha(\alpha + 1)$ is denoted as λ .

Notice p , p' , q , and r are all continuous on $[-1, 1]$. Also, p and r are both strictly positive on $(-1, 1)$ but are not both strictly positive at $x = \pm 1$.

5. Find the eigenvalues and corresponding eigenfunctions of each regular Sturm-Liouville eigenvalue problem. In (b) and (c), exploit the work you did on Homework 1; do not reproduce your work from Homework 1.

(a) $y'' + \lambda y = 0$, $0 < x < L$, $y'(0) = 0$, $y'(L) = 0$.

(b) $(e^{-6x}y')' + (1 + \lambda)e^{-6x}y = 0$, $0 < x < 8$, $y(0) = 0$, $y(8) = 0$.

(c) $\left(\frac{1}{x}y'\right)' + (4 + \lambda)x^{-3}y = 0$, $1 < x < e^4$, $y(1) = 0$, $y(e^4) = 0$.