# 3450:439/539:001 **Homework 4** Spring 2008

Recommended due date Thursday, February 7, 2008

**THIS HOMEWORK IS NOT FOR COURSE CREDIT.** However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

1. Consider the ODE

$$r^2R''(r) + rR'(r) + r^2\beta^2R(r) = 0, \quad 0 < r < a. \tag{1}$$

   (a) Make a change of variables to reexpress the ODE as a Bessel's equation.

   (b) Describe the values of $\beta$ for which (1) has a nontrivial solution, subject to the boundary conditions $R(a) = 0$ and $\lim_{r\to 0^+} |R(r)| < \infty$.

   (c) What is the nontrivial solution (up to an arbitrary constant multiple)?

2. Consider the ODE

$$r^2R''(r) + rR'(r) - r^2\alpha^2R(r) = 0, \quad 0 < r < a. \tag{2}$$

   (a) Make a change of variables to reexpress the ODE as a modified Bessel's equation.

   (b) Write the general solution to the ODE (2) in terms of modified Bessel functions.

   (c) Solve the ODE (2) subject to the boundary conditions as in Problem (1b).

3. Rodrigues's formula gives the $n$th Legendre polynomical $P_n(x)$, $n \in \mathsf{Z}^+$, via differentiation:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

   Use it to find $P_n(x)$, $n = 0, 1, 2, 3, 4$?

4. Consider **Legendre's equation** of "order $\alpha$"

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

   In this problem, we will find a series solution about the point $x = 1$.

   (a) Explain why $x = 1$ is a regular singular point. (Note $x = -1$ is also.)

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(b) Make the change of variables $t = x-1$. (To find a series solution about the point $x = 1$, we will find a series solution about the point $t = 0$.)

(c) Determine the indicial equation and its roots for the point $t = 0$.

(d) Find the first three terms of a series solution in powers of $t$ (and hence of $x - 1$). It turns out that for $n \geq 2$, the general term has the form

$$(-1)^{n+1}\frac{\alpha(\alpha+1)[2\cdot 1-\alpha(\alpha+1)]\cdots[n(n-1)-\alpha(\alpha+1)]}{2^n(n!)^2}(x-1)^n,$$

which gives you a check on the third term you found.