3450:439/539:001 **Homework 11** Spring 2008

Recommended due date: Friday, April 10, 2008

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

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1. Solve the initial-boundary-value problem for heat conduction:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2}, && 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, && t \geq 0, \\ u(x,0) &= f(x), && 0 \leq x \leq L, \end{aligned}$$

   for the case $f(x) = x(L - x)$. You may leave coefficients in integral form, or use a computer-algebra system (CAS) to integrate if you wish.

2. Solve by separation of variables

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, && 0 < x < L, \quad t > 0, \\ \frac{\partial u}{\partial x}(0,t) &= 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, && t \geq 0, \\ u(x,0) &= f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), && 0 \leq x \leq L. \end{aligned}$$

3. Separate variables via $u(x,t) = X(x)T(t)$ for the telegraph equation, which models transverse vibration in a uniform 1-D rod of length π as:

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < \pi, \quad t > 0.$$

   Here $u(x,t)$ is the vertical displacement at time t of the rod at horizontal position x, and a is a material parameter. Find values of the separation constant, and solve for X and T in the case of free ends:

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(\pi,t) = \frac{\partial^3 u}{\partial x^3}(0,t) = \frac{\partial^3 u}{\partial x^3}(\pi,t) = 0.$$

4. Let f be *any* twice differentiable function of one variable. Show that $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$ satisfies $y_{tt} = c^2 y_{xx}$.

(OVER)

5. Formulate a boundary-value problem (PDE, boundary and initial conditions) for vibration of a rectangular membrane occupying a region $0 \le x \le a$, $0 \le y \le b$ if the initial position is given by $f(x, y)$, and the initial velocity is given by $g(x, y)$. The membrane is fastened to a stiff frame along the rectangular boundary of the region.

6. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \quad -\infty < x < \infty.$$

   (a) Let
   $$f(x) = \begin{cases} x^2(1-x)^2, & 0 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases}$$
   and $g(x) \equiv 0$. Sketch $y = f(x)$ and $y = g(x)$, and interpret them physically (in one sentence) in the context of a vibrating string.

   (b) Solve the initial-value problem.

   (c) Sketch the solution for $t = 0, 1, 2$, and interpret each picture physically (in one sentence) in the context of a vibrating string. **For simplicity, take $c = 1$.**

7. Repeat Problem 6 above for $f(x) \equiv 0$ and $g(x) = \sin x$.