

# 3450:439/539:001 Homework 10 Spring 2008

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Optionally due: Thursday, April 3, 2008

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**THIS HOMEWORK MAY BE SUBMITTED FOR A GRADE IF YOU SO CHOOSE.** If you submit this homework, the grade will replace your lowest homework grade (provided it is higher than your lowest grade).

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1. Introducing *dimensionless variables* can make solving differential equations easier. Consider the one-dimensional undamped unforced wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

- (a) If  $u$  measures the vertical displacement of a string whose length at rest is  $L$ ,  $x$  is the horizontal direction, and  $t$  is time, what are the units on the parameter  $c$ ?
- (b) Show why the variable  $\xi = x/L$  is called a dimensionless variable.
- (c) Show that changing into the dimensionless variable  $\xi = x/L$  produces the equation

$$\frac{L^2}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2}, \quad 0 < \xi < 1, \quad t > 0.$$

- (d) Find the form of a constant  $k$  so that a change of variables  $\tau = kt$  produces the equation

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial \xi^2}, \quad 0 < \xi < 1, \quad \tau > 0.$$

(Note that the new formulation above does not explicitly involve any parameters. After solving, one can change back into the variables  $x$  and  $t$ , reintroducing the parameters  $c$  and  $L$ .)

- (e) Show that your variable  $\tau$  from (1d) is dimensionless.

2. Consider the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

This equation can be used to model heat conduction in a one-dimensional rod for positive time  $t$ , where  $x$  is position along the rod,  $u(x, t)$  is temperature of the rod at position  $x$  and time  $t$ , and  $\kappa$  is a parameter that depends on the material from which the rod is constructed. **Consider the case  $\kappa = 1$ .**

- (a) In one complete sentence, give a physical interpretation for the boundary conditions  $u(0, t) = 0$ ,  $u(L, t) = 0$ .
- (b) In one complete sentence, give a physical interpretation for the initial condition  $u(x, 0) = f(x)$ . Assume  $f(x)$  is a prescribed function.
- (c) Verify by substitution that  $u(x, t) = \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$  satisfies the PDE, as well as the boundary conditions in Problem (2a).
- (d) Show that  $u(x, t) = \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$  does not satisfy the initial condition in Problem (2b) in general, i.e. for arbitrary prescribed  $f(x)$ .
- (e) Does the supersposition  $u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$  satisfy the PDE? Does it satisfy the boundary conditions in Problem (2a)? Explain your answers in one complete sentence. (You do not need to mention convergence of the series; assume the series converges appropriately.)
- (f) Suppose the initial condition in Problem (2b) is prescribed with

$$f(x) = \sin\left(\frac{\pi}{L}x\right) - 3 \sin\left(\frac{4\pi}{L}x\right).$$

How should the constants in  $u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$  be chosen so that  $u(x, t)$  satisfies the initial condition?

- (g) Suppose the initial condition in Problem (2b) is prescribed with  $f(x) = 50$ . In one complete sentence, give a physical interpretation for the initial condition  $u(x, 0) = 50$ .
- (h) How should the constants in  $u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$  be chosen so that  $u(x, t)$  satisfies the initial condition  $u(x, 0) = 50$ ?

3. Prove that

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

for the vector field  $\mathbf{F} = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ . This is Green's second identity, which we use to derive a wave equation from four Maxwell's equations. The proof is a review of vector calculus.