3450:439/539:001 **Homework 1** Spring 2008

Recommended due date: Thursday, January 17, 2008

THIS HOMEWORK IS NOT FOR COURSE CREDIT. However, you need to do problems to learn the material. Also, about 1/3 of your exam will consist of recommended homework problems.

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1. Consider the ordinary differential equation (ODE) $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin(t)$.
   (a) Write the equation in the form $L(y) = f$. (Identify L and f.)
   (b) Is the equation linear? Prove or disprove using the definition of a linear operator.

2. Solve the initial-value problem (IVP) $\dfrac{dT_n}{dt} + \dfrac{n^2\pi^2 k}{L^2} T_n = B_n(t)$ $(t > 0)$, $T_n(0) = b_n$, where $B_n(t)$ is prescribed and so are the constants b_n, k, and L. Here $n = 1, 2, 3, \ldots$. Show the solution can be expressed as

$$T_n(t) = \int_0^t \exp\left(\frac{-n^2\pi^2 k(t-\tau)}{L^2}\right) B_n(\tau)\, d\tau + b_n \exp\left(\frac{-n^2\pi^2 k}{L^2} t\right).$$

3. Prove that the superposition principle holds for all linear homogeneous equations. That is, if u and v satisfy the linear homogeneous equation $L(x) = 0$ then $c_1 u + c_2 v$ also satisfies the equation for any numbers c_1 and c_2.

4. Referring to your proof in Problem 3, briefly explain why the superposition principle does not hold for nonlinear equations.

5. Referring to your proof in Problem 3, briefly explain why the superposition principle does not hold for nonhomogeneous equations.

6. Consider the ODE $au'' + bu' + cu = 0$ with $b^2 - 4ac < 0$. The system has two complex-valued solutions of the forms $u_1 = e^{(p+iq)x}$ and $u_2 = e^{(p-iq)x}$.
   (a) Reexpress u_1 and u_2 in the form $a+ib$ (using rules of exponents, Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, [redacted]).
   (b) Because the ODE is linear homogeneous, the superposition principle holds. Write down the solution $u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2$. Write down the solution $u_4 = \frac{1}{2i}u_1 - \frac{1}{2i}u_2$. Use these two (linearly independent) real-valued solutions u_3 and u_4 to write down a real-valued general solution to the system. (It's the same one we found in class.)

(OVER)

7. Solve the ODE $(e^{-6x}y')' + (1+\lambda)e^{-6x}y = 0$ for $y(x)$. Give three different forms corresponding to three different cases for λ.

8. Consider the ODE $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, $x > 0$, $\nu \geq 0$ (Bessel's equation of order ν). Reexpress the equation via the change of variables $x = \lambda z$.

9. Show the partial differential equation (PDE) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + A\dfrac{\partial u}{\partial x} + Bu$ ~~$= 0$~~ can be transformed into $\dfrac{\partial v}{\partial t} = k\dfrac{\partial^2 v}{\partial x^2}$ by making a change of variables $u(x,t) = \exp(\alpha x + \beta t)v(x,t)$. What values of α and β must be used?

10. Solve the ODE $\left(\dfrac{1}{x}y'\right)' + (4+\lambda)x^{-3}y = 0$ for $y(x)$, $x > 0$. Give three different forms corresponding to three different cases for λ.