1. Consider the partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

Laura Gross

This equation can be used to model vibrations of a string for positive time t, where x is horizontal position along along the string, $u(x,t)$ is the vertical displacement of the string at position x and time t, and c is a parameter that depends on the material from which the string is constructed.

(a) In one complete sentence, give a physical interpretation for the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$.

The ends of the string are anchored at $x = 0$ & $x = L$ with no vertical displacement for all time.

(b) In complete sentences, give a physical interpretation for the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Assume $f(x)$ and $g(x)$ are prescribed functions.

The initial deflection at x is $f(x)$, & the initial velocity at x is $g(x)$.

(c) Verify by substitution that $u(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right)$ satisfies the PDE, as well as the boundary conditions in Problem (1a) for $n = 1, 2, 3, \ldots$. **(DO NOT DERIVE THE SOLUTIONS; ONLY VERIFY BY SUBSTITUTION.)**

$$u_{tt} = -\left(\frac{cn\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L} = c^2 u_{xx} \checkmark$$

$$u(0,t) = \sin(0)\cos\frac{cn\pi t}{L} = 0$$

$$u(L,t) = \sin(n\pi)\cos\frac{cn\pi t}{L} = 0, \quad n = 1, 2, 3, \ldots$$

(d) Does the supersposition $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn^2\pi^2 t}{L^2}\right)$ satisfy the PDE? — yes. Does it satisfy the boundary conditions in Problem (1a)? — yes. ~~Give~~ a full explanation for your answers in one complete sentence. (You do not need to mention convergence of the series; assume the series converges appropriately.)

The PDE & BCs are linear & homogeneous, so the superposition satisfies them.

(OVER)

2. Solve

$$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2} + x + t, \qquad -\infty < x < \infty, \;\; t > 0,$$

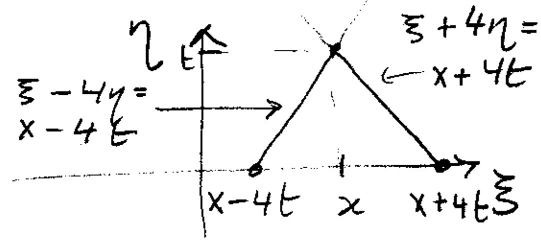
Laura Gross

$$u(x,0) = x, \;\; \frac{\partial u}{\partial t}(x,0) = e^{-x}, \;\; -\infty < x < \infty. \qquad c = 4$$

(a) $\frac{1}{2}\left(f(x-ct) + f(x+ct)\right) = \frac{1}{2}\left(f(x-4t) + f(x+4t)\right) = \frac{1}{2}\left[x - 4t + x + 4t \right] = \underline{\underline{x}}$

(b) $\frac{1}{2c}\int_{x-ct}^{x+ct} e^{-\tau}\,d\tau = \frac{1}{8}\left[-e^{-(x+4t)} + e^{-x+4t}\right]$

$= \underline{\underline{\frac{1}{8}\left[e^{-x+4t} - e^{-x-4t}\right]}}$



(c) $\frac{1}{2c}\iint_{\Delta} (\xi + \eta)\, dA$

$= \frac{1}{8}\int_0^t \int_{x+4(\eta-t)}^{x-4(\eta-t)} (\xi+\eta)\, d\xi\, d\eta = \frac{1}{8}\int_0^t \left(\frac{\xi^2}{2} + \eta\xi \right)\Big|_{x+4(\eta-t)}^{x-4(\eta-t)} d\eta$

$= \frac{1}{8}\int_0^t \left\{ \frac{(x-4(\eta-t))^2 - (x+4(\eta-t))^2}{2} + \eta\left[\cancel{x} - 4(\eta-t) - \cancel{x} - 4(\eta-t)\right]\right\} d\eta$

$= \cancel{\frac{1}{8}}\int_0^t \left\{ \frac{-\cancel{8}(\eta-t)x\cdot\cancel{2}}{\cancel{2}} - \cancel{8}\eta^2 + \cancel{8}t\eta \right\} d\eta$

$= \int_0^t \left\{ \eta(-x+t) - \eta^2 + tx \right\} d\eta$

$= \left(-\frac{\eta^3}{3} + (t-x)\frac{\eta^2}{2} + tx\eta \right)\Big|_{\eta=0}^{t} = -\frac{t^3}{3} + \frac{t^3}{2} - \frac{xt^2}{2} + xt^2$

$= \underline{\underline{\frac{t^3}{6} - \frac{xt^2}{2}}}$

$u(x,t) = (a) + (b) + (c) = \boxed{x + \frac{1}{8}\left[e^{-x+4t} - e^{-x-4t}\right] + \frac{t^3}{6} - \frac{xt^2}{2}}$

3. Solve

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2, \quad t > 0,$$
$$u(0,t) = T_1, \quad u(2,t) = T_2, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 < x < 2.$$

$u(x,t) = v(x,t) + \psi(x)$

$v_t = k(v_{xx} + \psi''(x))$

$v(0,t) + \psi(0) = T_1$

$v(2,t) + \psi(2) = T_2$

$v(x,0) + \psi(x) = f(x)$

$\cancel{k}\psi'' = 0$

$\psi(0) = T_1 \quad \psi(2) = T_2$

$\psi = ax + b$

$\psi(0) = b = T_1$

$\psi(x) = ax + T_1$

$\psi(2) = a2 + T_1 = T_2 \Rightarrow$

$a = \frac{T_2 - T_1}{2}$

$\psi(x) = \left(\frac{T_2 - T_1}{2}\right)x + T_1$

$v_t = k v_{xx}$

$v(0,t) = 0, \; v(2,t) = 0$

$v(x,0) = f(x) - \psi(x)$

$\downarrow$

from handout & subbing in ψ,
get series for $v(x,t)$.

$$\boxed{\begin{aligned} u(x,t) &= \left(\frac{T_2 - T_1}{2}\right)x + T_1 + \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2}\, e^{-\left(\frac{n\pi}{2}\right)^2 kt}, \\ c_n &= \frac{2}{2}\int_0^2 \left[f(x) - \left(\frac{T_2 - T_1}{2}\right)x - T_1\right] \sin\frac{n\pi x}{2}\, dx. \end{aligned}}$$

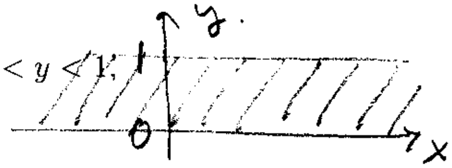
Laura Gross

4. Solve the following boundary-value problem using the method of Fourier transform.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < x < \infty, \ 0 < y < 1,$$

Laura Gross

$$u(x,0) = 0, \ \ u(x,1) = f(x), \ \ -\infty < x < \infty.$$



Sketch the problem domain. **Show clearly what facts and conditions you use to treat boundary terms when you integrate by parts.** Present the solution as a convolution integral. Express inverse transform(s) in integral form.

$$\int_{-\infty}^{\infty} e^{-i\omega x} \underbrace{\frac{\partial^2 u}{\partial x^2} dx}_{dv} = -\int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial^2 u}{\partial y^2} dx.$$

$$e^{-i\omega x} \frac{\partial u}{\partial x}\Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} e^{-i\omega x} \underbrace{\frac{\partial u}{\partial x} dx}_{dv} =$$

$$-\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(e^{-i\omega x} u\right) dx.$$

Assuming $\lim_{x \to \pm\infty} \frac{\partial u}{\partial x} = 0$, boundary terms are 0.

$$i\omega \left[e^{-i\omega x} u \Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} e^{-i\omega x} u \, dx \right] =$$

$$-\frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} e^{-i\omega x} u \, dx.$$

Assuming $\lim_{x \to \pm\infty} u = 0$, boundary terms are 0.

$$+\omega^2 \hat{u}(\omega, y) = +\frac{\partial^2 \hat{u}}{\partial y^2}(\omega, y) \qquad \hat{u}(\omega, y) = e^{ry} \rightarrow$$

$$\omega^2 = r^2 \rightarrow r = \pm\omega \rightarrow \hat{u} = c_1 \cosh \omega y + c_2 \sinh \omega y$$

$$\int_{-\infty}^{\infty} e^{-i\omega x} u(x,0) dx = \int_{-\infty}^{\infty} e^{-i\omega x} 0 \, dx \rightarrow \hat{u}(\omega, 0) = 0$$

Similarly $\hat{u}(\omega, 1) = \hat{f}(\omega)$.

$$\hat{u}(\omega, 0) = c_1 = 0 \rightarrow \hat{u} = c_2 \sinh \omega y$$

$$\hat{u}(\omega, 1) = c_2 \sinh \omega = \hat{f}(\omega) \rightarrow c_2 = \frac{\hat{f}(\omega)}{\sinh \omega}.$$

$$\hat{u}(\omega, y) = \hat{f}(\omega) \frac{\sinh \omega y}{\sinh \omega} = \hat{h}(\omega, y)$$

(OVER)

$$u(x,y) = f(x) * h(x,y) = \boxed{\int_{-\infty}^{\infty} f(\xi) h(x-\xi, y) d\xi = u(x,y)}$$

$$h(x,y) = \mathcal{F}^{-1}\{\hat{h}(\omega, y)\} = \boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{h}(\omega, y) d\omega = h(x,y)}$$

5. Solve

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0<r<1, \quad t>0,$$
$$\lim_{r\to 0^+} |u(r,t)| < \infty, \quad u(1,t)=0, \qquad t>0,$$
$$u(r,0) = f(r), \qquad 0<r<1.$$

Laura Gross

Sketch the problem domain. Note this problem can model heat conduction on a disk of radius 1 in which the heat conduction has no angular dependence but depends only on distance from the center. **Assume that the eigenvalue problem has only nonzero eigenvalues and that the eigenvalues all have the same sign. When you solve the eigenvalue problem, show how you apply the boundary conditions.**

$$u(r,t) = R(r)T(t)$$

$$\frac{RT' = \kappa\left(R''T + \frac{1}{r}R'T\right)}{\kappa RT} = \delta$$

$$\frac{T'}{\kappa T} = \frac{R'' + \frac{1}{r}R'}{R} = \delta$$

$$\begin{cases} R'' + \frac{1}{r}R' - \delta R = 0 \\ \lim_{r\to 0^+} |R(r)| < \infty, \; R(1) = 0 \end{cases}, \qquad T' - \delta\kappa T = 0$$

$$T = e^{pt} \to p = \delta\kappa$$

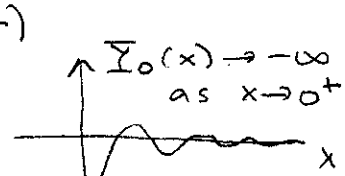
let $\delta = -\lambda^2 < 0 \quad (\lambda > 0)$

$$T_n(t) = e^{-\beta_n^2 \kappa t}$$

$$(rR')' + \lambda^2 r R = 0$$

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r)$$

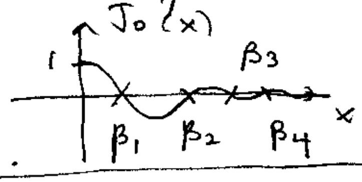
$Y_0(x) \to -\infty$ as $x \to 0^+$



$$\lim_{r\to 0^+} |R(r)| < \infty \to c_2 = 0$$

$$R(r) = c_1 J_0(\lambda r)$$

$R(1) = c_1 J_0(\lambda) = 0$ if



$$\sqrt{-\delta} = \lambda = \beta_n, \; n = 1, 2, 3, \ldots$$

$$R_n(r) = J_0(\beta_n r)$$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\beta_n r)\, e^{-\beta_n^2 \kappa t}}$$

$$u(r,0) = \sum_{n=1}^{\infty} c_n J_0(\beta_n r) = f(r) \Rightarrow \boxed{c_n = \frac{\int_0^1 f(r) J_0(\beta_n r)\, r\, dr}{\int_0^1 [J_0(\beta_n r)]^2\, r\, dr}}$$