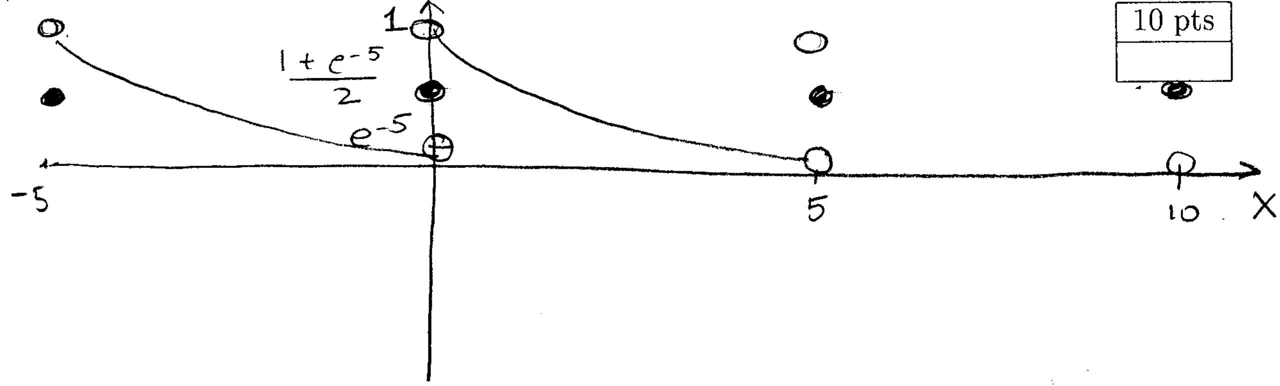
Laura Hross

1. Consider the function f that has period $2L = 5$ and $f(x) = e^{-x}$ for $0 \leq x < 5$.

(a) Graph the function to which its Fourier series converges. Include three periods.

10 pts



(b) Find the **complex form** of the Fourier series expansion of $f(x)$. Evaluate the integrals. Remember to find the zeroth coefficient. $L = 5/2$

15 pts

$$\sum_{-\infty}^{\infty} c_n e^{-2in\pi x/5} \quad \text{, where}$$

$$c_n = \frac{1}{5}\int_0^5 e^{-x} e^{2in\pi x/5}\, dx = \frac{1}{5}\int_0^5 e^{x(2in\pi/5 - 1)}\, dx$$

$$= \frac{1}{5} \frac{1}{\frac{2in\pi}{5} - 1} e^{x\left(\frac{2in\pi}{5} - 1\right)} \Big|_0^5$$

$$c_n = \frac{1}{2in\pi - 5}\left(e^{2in\pi - 5} - 1\right) \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$= \frac{1}{2in\pi - 5}\left(e^{-5}[\cos 2n\pi + i \sin 2n\pi] - 1\right)$$

$$= \frac{1}{2in\pi - 5}\left(e^{-5} - 1\right)$$

Check $\quad c_0 = \frac{1}{5}\int_0^5 e^{-x}\, dx = -\frac{1}{5} e^{-x} \Big|_0^5$

$$= -\frac{1}{5}\left(e^{-5} - 1\right) \quad \checkmark$$

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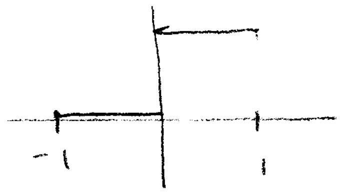


2. Consider the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$

Find the **phase angle form** of the Fourier series. Evaluate the integrals.

15 pts

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin(n\pi x + \phi_n).$$

$$A_n = \sqrt{a_n^2 + b_n^2} \qquad \tan\phi_n = \frac{a_n}{b_n}$$

$$a_0 = \frac{1}{1}\left[\int_{-1}^{0} 0\,dx + \int_0^1 1\,dx\right] = x\Big|_0^1 = 1 = a_0$$

$$a_n = \frac{1}{1}\left[\int_0^1 \cos n\pi x\,dx\right] = 0 \rightarrow \text{F. S. is } \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

This is phase-angle form if $b_n \geq 0$

$$b_n = \frac{1}{1}\left[\int_0^1 \sin n\pi x\,dx\right] = -\frac{\cos n\pi x}{n\pi}\Big|_0^1$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ \frac{2}{n\pi}, & n \text{ odd} \end{cases}$$

$$A_n = b_n = \frac{1}{n\pi}\left[1 - (-1)^n\right].$$

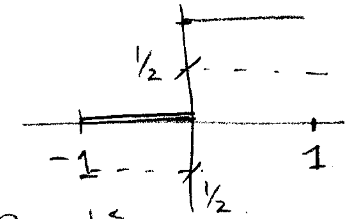
$$\phi_n = 0$$

(OVER)

alternative

2. Consider the function

$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$



Find the **phase angle form** of the Fourier series. Eval the ints.

15 pts

shifting down $\frac{1}{2}$ unit gives an odd fnc. w/ F.S.S. expan. Expand it, & shift back up

$$(*) \quad + \frac{1}{2} + \sum_{1}^{\infty} b_n \sin n\pi x$$

← phase-angle form if $b_n \geq 0$ ($A_n = b_n$, $\phi_n = 0$).

$$b_n = \frac{2}{1} \int_0^1 \tfrac{1}{2} \sin n\pi x \, dx$$

$$= \left. \frac{-\cos n\pi x}{n\pi} \right|_0^1$$

$$b_n = -\frac{1}{n\pi} \left[(-1)^n - 1 \right] = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

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(OVER)

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3. Consider the eigenvalue problem $x^2y''(x) + xy'(x) + x^2\lambda y(x) = 0$, $0 < x < 1$, $y(1) = 0$ and $\lim_{x\to 0^+} |y(x)| < \infty$.

x

(a) Assuming that the boundary condition at zero is "appropriate," show that the problem is a singular Sturm-Liouville boundary-value problem.

10 pts

$$xy'' + y' + x\lambda y = 0.$$

$$(xy')' + \lambda xy = 0. \iff (py')' + (-qy + \lambda ry) = 0,$$

where $p = x$, $q = 0$, $r = x$.

(i) p, p', q, r are cont. on $[0,1]$.

(ii) $p, r > 0$ on $(0,1]$, but $p(0) = 0$, $r(0) = 0$.

The b.c. at each end has an appropriate form.

All the eigenfncs for this prob are pos.

(b) ~~Examine the cases of λ positive, negative, and zero, and~~ find the eigenvalues and eigenfunctions. Put boxes around the eigenvalues and eigenfunctions.

~~15 pts~~	10.

$$\lambda = \beta^2$$

$$x^2y'' + xy' + \beta^2x^2y = 0.$$

$$\rightarrow \quad y = c_1 J_0(\beta x) + c_2 Y_0(\beta x)$$

BC $\lim_{x\to 0^+} |y| < \infty \rightarrow c_2 = 0$ bec. $\lim_{x\to 0^+} Y_0 = -\infty$.

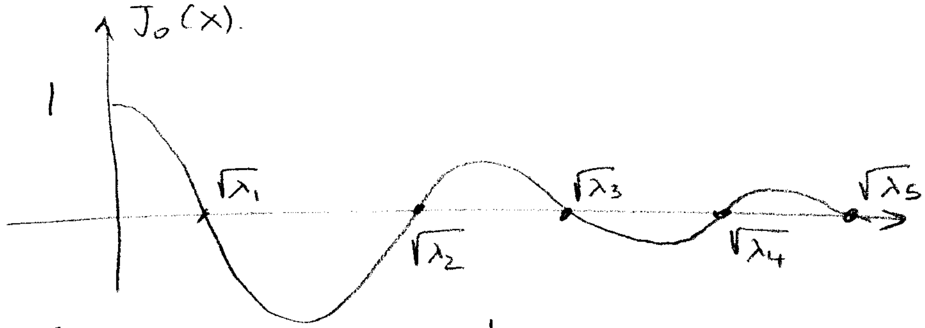
$$y = c_1 J_0(\beta x).$$

BC $y(1) = c_1 J_0(\beta) = 0 \rightarrow$ choose β_n = n^{th} root of $J_0(x)$. b_n

$$\boxed{\lambda_n = \beta_n^2 = b_n^2, \quad y_n(x) = J_0(b_n x)}$$

(c) Show that eigenvalues you found in Problem (3b) have the properties guaranteed by the Sturm-Liouville Theorem via a quick sketch of a graph.

10 pts



(a) $\sqrt{\lambda_i}$ are real

(b) There are infinitely many $\sqrt{\lambda_i}$.

(c) $\sqrt{\lambda_1} < \sqrt{\lambda_2} < \cdots$

(d) $\lim_{i \to \infty} \sqrt{\lambda_i} = \infty$

(d) Find the form of an eigenfunction expansions of a function $f(x)$ (where $f(x)$ is is piecewise smooth on the set of all real numbers) using the eigenfunctions of Problem (3b). Show the forms of the coefficients using inner products, and define each inner product $\langle g, h\rangle$ in terms of an integral. **(Do not evaluate integrals; just set them up.)** On what interval is each expansion valid?

10 pts

$$\sum_{n=1}^{\infty} c_n J_0(b_n x)$$

$$c_n = \frac{\langle f(x), J_0(b_n x)\rangle}{\langle J_0(b_n x), J_0(b_n x)\rangle} = \frac{\int_0^1 f(x) J_0(b_n x)\, x\, dx}{\int_0^1 J_0^2(b_n x)\, x\, dx}.$$

$0 < x < 1$.

Laura Gross

(OVER)

Laura Hiose

4. Consider the regular Sturm-Liouville problem $y'' + \lambda y = 0$ on $0 < x < 1$, $\alpha y(0) + y'(0) = 0$, $y(1) = 0$, where α is a given real constant. Show $\alpha = 1$ is the only α value for which $\lambda = 0$ is an eigenvalue.

10 pts

$$y'' = 0 \rightarrow y' = c_1$$

$$y = c_1 x + c_2.$$

$$y(1) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1. \quad (*)$$

$$y = c_1 x - c_1$$

$$\alpha y(0) + y'(0) = \alpha(-c_1) + c_1 = 0$$

$$\Leftrightarrow c_1(1-\alpha) = 0$$

$$\Rightarrow \underbrace{c_1 = 0}_{\downarrow} \quad \text{or} \quad \underbrace{\alpha = 1}_{\downarrow}.$$

$c_2 = 0$ by $(*)$ $\qquad y(x) = c_1(x-1) \not\equiv 0$

$y(x) \equiv 0 \rightarrow \lambda = 0$ not an eigenvalue

So $\alpha = 1$ is the only α value for which $\lambda = 0$ is an eigenvalue.

5. Show that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ using the definition of the Laplace transform and integration by parts. (You may assume that $f(t)$ and $f'(t)$ are piecewise continuous on $0 \le t < \infty$ and of exponential order.)

10 pts

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$$

$$= e^{-st} f(t) \Big|_0^\infty + s\int_0^\infty f(t) e^{-st}\,dt$$

$$= \underbrace{\lim_{c\to\infty} (e^{-st} f(t))}_{=0 \text{ bec. } f(t) \text{ is of exponential order.}} - f(0) + sF(s). \checkmark$$