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1. Consider the PDE $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + A\frac{\partial u}{\partial x} + Bu$, where k, A, and B are parameters.

Does the superposition principle hold for this equation? (Circle one:) YES NO

(That is, if $v(x,t)$ and $w(x,t)$ are two solutions to the equation, does $c_1 v(x,t) + c_2 w(x,t)$ necessarily satisfy the equation for all constants c_1 and c_2?)

Explain your answer in one complete sentence.

It is a homogeneous linear equation.

10 pts

2. Solve the ODE $(x^3 y')' + \lambda x y = 0$ for $y(x)$, $x > 0$. (Give three different forms of the general solution corresponding to three different cases for the arbitrary parameter λ.)

15 pts

$$\frac{3x^2 y' + x^3 y'' + \lambda x y = 0}{x}$$

$x^2 y'' + 3xy' + \lambda y = 0$. $y = x^m \rightarrow$.

$m(m-1) + 3m + \lambda = 0 \rightarrow m^2 + 2m + \lambda = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}.$$

(I) $1-\lambda > 0$ $(\lambda < 1)$: $y = c_1 x^{(-1+\sqrt{1-\lambda})} + c_2 x^{(-1-\sqrt{1-\lambda})}$

(II) $\lambda = 1$: $y = c_1 x^{-1} + c_2 x^{-1} \ln x$.

(III) $\lambda > 1$: $y = c_1 x^{-1} \cos(\sqrt{\lambda - 1} \ln x) + c_2 x^{-1} \sin(\sqrt{\lambda-1} \ln x)$

3. Consider Legendre's equation of order α:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad -1 < x < 1.$$

Rodrigues's formula gives the nth Legendre polynomical $P_n(x)$, $n \in \{0, 1, 2, \ldots\}$:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n.$$

10 pts

(a) Use Rodrigues's formula to find $P_1(x)$.

$$\frac{1}{2^1(1)} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2}[2x] = \boxed{x. = P_1(x)}$$

(b) Verify by substitution that the function $P_1(x)$ you found in Problem (3a) satisfies Legendre's equation (above) of order $\alpha = 1$.

$$(1-x^2)P_1'' - 2xP_1' + (1)(2)P_1 \stackrel{?}{=} 0.$$

$$P_1' = 1, \quad P_1'' = 0.$$

$$-2x(1) + 2x = 0 \checkmark$$

4. Suppose that $f(x)$ can be expanded into a series of the form

$$\sin\left(\frac{\pi}{L}x\right) - 3\sin\left(\frac{4\pi}{L}x\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right). = b_1 \sin\left(\frac{\pi}{L}x\right) + b_2 \sin\left(\frac{2\pi}{L}x\right) + b_3 \sin\left(\frac{3\pi}{L}x\right) + \ldots$$

10 pts

If $f(x) = \sin\left(\frac{\pi}{L}x\right) - 3\sin\left(\frac{4\pi}{L}x\right)$, what are the values of b_n, $n = 1, 2, \ldots$?

$b_1 = 1$, $b_4 = -3$.

$b_n = 0$, $n = 2, 3, 5, 6, 7, 8, \ldots$

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5. Consider the ODE $4xy'' + 2y' + y = 0$.

(a) Show that $x = 0$ is a regular singular point.

10 pts

$\frac{Q}{P} = \frac{2}{4x}$, $\frac{R}{P} = \frac{1}{4x}$ } not both analytic at $x=0$

$x\frac{Q}{P} = \frac{1}{2}$, $x^2\frac{R}{P} = \frac{x}{4}$ } both analytic at $x=0$.

(b) Assume a Frobenius solution about the point $x = 0$. Find and solve the indicial equation.

10 pts

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad n = m-1$

$2y' = \sum_{n=0}^{\infty} 2a_n (n+r) x^{n+r-1}$ $\quad n=m$

$4xy'' = \sum_{n=0}^{\infty} 4a_n (n+r) x^{n+r-1} (n+r-1)$ $\quad n=m$.

$y = \sum_{m=1}^{\infty} a_{m-1} x^{m+r-1}$

$m = 0$: $2a_0 (r + 2r(r-1)) = 0$

$a_0 \neq 0 \rightarrow$ $2r^2 + r - 2r = 0$

$2r^2 - r = 0$

$\boxed{r(2r-1) = 0}$

$\boxed{r = 0 \text{ or } r = \frac{1}{2}}$

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(c) Determine a recurrence relation. Be sure to indicate the values of the index for which it is valid.

10 pts

$\boxed{m \geq 1}$: $a_m 2(m+r)\left[1 + 2(m+r-1)\right] = -a_{m-1}$.

$$\boxed{a_m = \frac{-1}{2(m+r)(2m+2r-1)} a_{m-1}}$$

$$\boxed{\sum_{n=0}^{\infty} a_n x^{n+0}, \quad \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}}$$

SOLUTION A

(d) State the **forms** of two linearly independent Frobenius solutions. Write out the first few terms of **one** Frobenius solution, making sure to apply your recurrence relation a couple of times. (You need **not** do any arithmetic simplification.) Do not find the coefficients for the second solution.

10 pts

$r = 0$: $a_m = \frac{-1}{2m(2m-1)} a_{m-1}$

$m = 1$: $a_1 = \frac{-1}{2(1)} a_0 = -\frac{1}{2!} a_0$

$m = 2$: $a_2 = \frac{-1}{2(2)(3)} a_1 = \frac{-1}{4 \cdot 3}\left(\frac{-1}{2 \cdot 1}\right) a_0 = \frac{1}{4!} a_0$.

$a_0 + a_1 x + a_2 x^2 + \cdots$

$= a_0 \boxed{\left[1 - \frac{1}{2!} x + \frac{1}{4!} x^2 + \cdots + \frac{(-1)^k}{(2k)!} x^k + \cdots\right.}$

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(c) Determine a recurrence relation. Be sure to indicate the values of the index for which it is valid.

10 pts

$m \geq 1: \quad b_m 2(m+r)\,[1 + 2(m+r-1)] = -b_{m-1}$

$$\boxed{b_m = \frac{-1}{2(m+r)(2m+2r-1)}\, b_{m-1}}$$

$$\boxed{\sum_{n=0}^{\infty} a_n x^{n+0}, \quad \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}}$$

SOLUTION B

(d) State the **forms** of two linearly independent Frobenius solutions. Write out the first few terms of **one** Frobenius solution, making sure to apply your recurrence relation a couple of times. (You need **not** do any arithmetic simplification.) Do not find the coefficients for the second solution.

10 pts

$r = \frac{1}{2}: \quad b_m = \frac{-1}{2(m+\frac{1}{2})(2m+1-1)}\, b_{m-1}$

$$= \frac{-1}{(2m+1)(2m)}\, b_{m-1}$$

$m = 1: \quad b_1 = \frac{-1}{3(2)} b_0 = \frac{-1}{3!} b_0$

$m = 2: \quad b_2 = \frac{-1}{5(4)} b_1 = \frac{-1}{5 \cdot 4}\left(\frac{-1}{3 \cdot 2}\right) b_0 = \frac{1}{5!} b_0$

$$\sum_{n=0}^{\infty} b_n x^n x^{1/2} = x^{1/2}\left[b_0 + b_1 x + b_2 x^2 + \cdots\right]$$

$$\boxed{= b_0 x^{1/2}\left[1 - \frac{1}{3!}x + \frac{1}{5!}x^2 + \cdots + (-1)^k \frac{1}{(2k+1)!} x^k + \cdots\right]}$$

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6. Consider the ODE

$$r^2R''(r) + rR'(r) + r^2\beta^2R(r) = 0, \quad 0 < r < a.$$

15 pts

(a) Make a change of variables to reexpress the ODE as a Bessel's equation.

$r\beta = x \rightarrow r = \frac{1}{\beta}x \qquad 0 < \frac{1}{\beta}x < a$

$R'(r) = \frac{d\hat{R}}{dx}\frac{dx}{dr} = \beta\frac{d\hat{R}}{dx}$

$R''(r) = \frac{d}{dx}\left(\beta\frac{d\hat{R}}{dx}\right)\frac{dx}{dr} = \beta^2\frac{d^2\hat{R}}{dx^2}$

$(\frac{1}{\beta}x)^2\beta^2\frac{d^2\hat{R}}{dx^2} + (\frac{1}{\beta}x)\beta\frac{d\hat{R}}{dx} + (\frac{1}{\beta}x)^2\beta^2\hat{R} = 0.$

$x^2\frac{d^2\hat{R}}{dx^2} + x\frac{d\hat{R}}{dx} + x^2\hat{R} = 0., \quad 0 < x < \beta a.$

(b) Write the general solution $R(r)$ in terms of Bessel functions.

$\hat{R}(x) = c_1 J_0(x) + c_2 Y_0(x)$

$R(r) = c_1 J_0(r\beta) + c_2 Y_0(r\beta), \quad 0 < r < a.$