Hw 9 – A soln

ACM II
Gross
S2008

(I) $f(x) = 1 - x, \quad 0 \le x < 2L$

$f(x+2L) = f(x)$.

(b). per $p = 2L$, $\quad \omega = \frac{2\pi}{2L} = \frac{\pi}{L}$

$$\sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \quad c_n = \frac{1}{2L} \int_{-L}^{L} \underbrace{f(x)}_{\text{per } 2L} \underbrace{e^{in\pi x/L}}_{\text{per } \frac{2\pi}{n\pi} \cdot L} dx \quad (*)$$

$$c_n = \frac{1}{2L} \Big\{ (1-x) e^{in\pi x/L} \left(\frac{L}{n\pi i}\right)\Big]_0^{2L} + \int_0^{2L} e^{in\pi x/L} \left(\frac{L}{n\pi i}\right) dx \Big\} = \qquad n \neq 0$$

$$= \frac{1}{2L} \Big\{ \left(-\frac{Li}{n\pi}\right)\left[(1-2L)e^{2in\pi} - 1\right] + \left(\frac{L}{n\pi i}\right)^2 e^{in\pi x/L}\Big]_0^{2L} \Big\}$$

$\Big[e^{2in\pi} = \cos 2n\pi + i \sin 2n\pi = 1.$

$$= \frac{1}{2L} \Big\{ -\frac{Li}{n\pi} \cdot (1 - 2L - 1) + \frac{L^2}{n^2\pi^2} \Big[\underbrace{e^{in\pi 2} - 1}_{0} \Big] \Big\}$$

$$= \frac{i}{\cancel{2}n\pi} (+\cancel{2}L) = \frac{+iL}{n\pi} = c_n, \; n \neq 0$$

$$\text{F.S.}: \quad \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{Li}{n\pi} e^{-in\pi x/L}$$

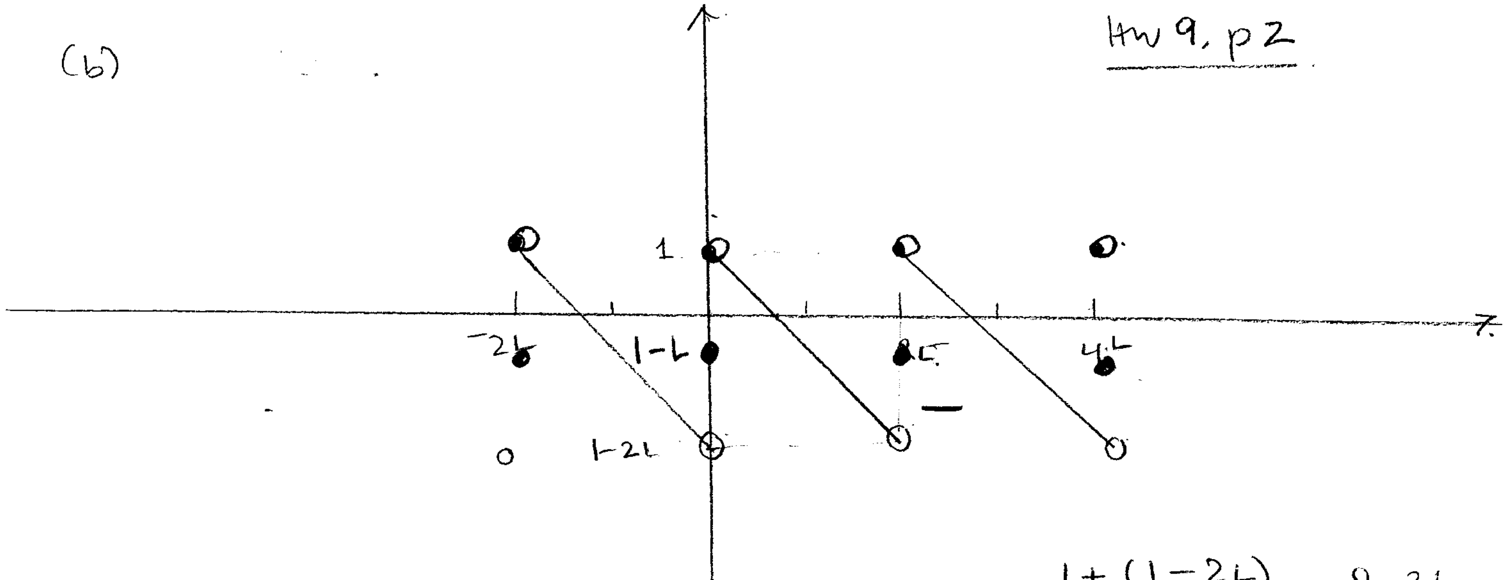
$$a_0 = \frac{1}{L} \int_0^{2L} (1-x)\, dx = \frac{1}{L}\left(x - \frac{x^2}{2}\right)\Big]_0^{2L} =$$

$$= \frac{1}{\cancel{L}}\left\{2\cancel{L} - \frac{4L^{\cancel{2}}}{2}\right\} = \frac{4(1-L)}{2} = 2(1-L)$$

$$\text{F.S.} \quad \boxed{1 - L + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{+Li}{n\pi} e^{-in\pi x/L}}$$

(*) Because the integrand has period 2L, we can equivalently integrate over any interval of length 2L, e.g. $0 < x < 2L$.

(b)



$$\frac{1+(1-2L)}{2} = \frac{2-2L}{2} = 1-L$$

$$1-L+\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{+iL}{n} e^{-inx\pi/L}$$

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(2). $\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\underbrace{e^{-i\omega x}}_{u}\,dx$

$$= e^{-i\omega x} f(x)\Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx$$

$\lim_{x\to\pm\infty} f(x)=0$ if f is absolutely integrable.

$$= i\omega\,\hat{f}(\omega)$$ ✓

(3). $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x}\, dx$

$$= \int_{-\infty}^{0} e^{x} e^{-i\omega x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{x(1-i\omega)}\, dx + \int_{0}^{\infty} e^{-x(1+i\omega)}\, dx$$

$$= \frac{1}{1-i\omega} e^{(1-i\omega)x}\Big|_{-\infty}^{0} + \frac{1}{-1-i\omega} e^{-(1+i\omega)x}\Big|_{0}^{\infty}$$

$$= \frac{1}{1-i\omega}\left(1 - \cancel{\lim_{N\to\infty} e^{-(1-i\omega)N}}\to 0^{*}\right) +$$

$$\frac{-1}{+1+i\omega}\left(\cancel{\lim_{N\to\infty} e^{-(1+i\omega)N}}\to 0^{*} - 1\right)$$

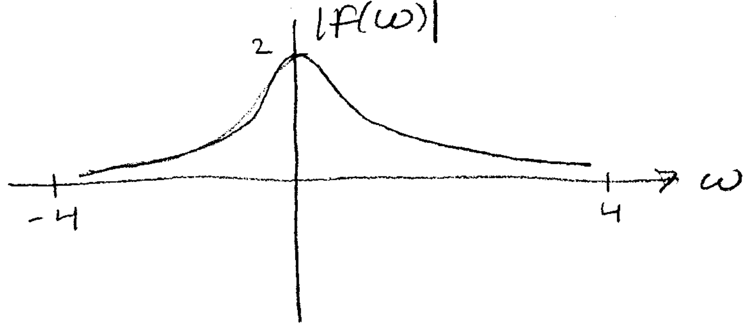
$$^{*}\ e^{-(1\mp i\omega)N} = e^{-N \pm i\omega N} = e^{-N}\left(\cos(\omega N) \pm i\sin(\omega N)\right)$$

constant-amplitude oscillations

decays as $N \to \infty$

$$= \frac{1\,(1+i\omega)}{1-i\omega\,(1+i\omega)} + \frac{1}{(1+i\omega)(1-i\omega)}\,(1-i\omega)$$

$$= \frac{1+i\omega+1-i\omega}{1+\omega^2} = \frac{2}{1+\omega^2} = \hat{f}(\omega).$$



(4.) $f(x) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 \le x < \infty \end{cases}$ (a) F.T. does not exist: $\int_{-\infty}^{\infty} |f(x)|dx = \infty$.

what if we try?

$$\Rightarrow \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi = \int_{0}^{\infty} e^{-i\omega\xi} d\xi$$

$$= \frac{e^{-i\omega\xi}}{-i\omega}\Big|_0^{\infty} = \frac{-1}{i\omega}\left[\lim_{N\to\infty} e^{-i\omega N} - 1\right]$$

$$= \frac{-1}{i\omega}\left(\lim_{N\to\infty} (\underset{\uparrow}{\cos \omega N} - i \underset{\uparrow}{\sin \omega N} - 1\right)$$

oscillate

Limit does not exist.

$\therefore$ F.T. does not exist.

(b) $$\mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-s\tau} f(\tau)\, d\tau = \int_0^{\infty} e^{-s\tau} d\tau$$

$$= \frac{e^{-s\tau}}{-s}\Big|_0^{\infty} = +\frac{1}{s}$$ Laplace transform

of $\tilde{f}(x) = 1$ on $0 \le x < \infty$ exists & equals $\frac{1}{s}$.

for $s > 0$.

($f(x)$ is piecewise cont. on $[0, \infty)$ & is of exponential order)

(5). $\mathcal{L}\{-3\cos 2t + 5\sin 4t\}$

$$= -3\,\mathcal{L}\{\cos 2t\} + 5\,\mathcal{L}\{\sin 4t\}$$

$$= -3\,\frac{s}{s^2+4} + \frac{5\cdot 4}{s^2+16}$$

$$= \frac{-3s}{s^2+4} + \frac{20}{s^2+16}$$

(6). $\mathcal{L}^{-1}\left\{\frac{2}{s^4}\left(\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^6}\right)\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^5} - \frac{6}{s^6} + \frac{8}{s^{10}}\right\}$$

$$= \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{6}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} + \frac{8}{9!}\mathcal{L}^{-1}\left\{\frac{9!}{s^{10}}\right\}$$

$$= \frac{2}{4!}t^4 - \frac{6}{5!}t^5 + \frac{8}{9!}t^9.$$

$$= \frac{1}{12}t^4 - \frac{1}{20}t^5 + \frac{1}{45{,}360}t^9$$