HW 8 – A sol'n

AEM II
Gross
Sp2008

(1) (a). f odd, g even; show fg odd:

$fg(-x) = f(-x)g(-x)$ by def of mult'n of fnc's.

$= [-f(x)]\, g(x)$ bec f odd & g even.

$= -fg(x)$. So fg is odd by def ✓

fg is odd by the def. of odd.

(b). h odd. $\int_{-L}^{L} h(x)\,dx = \int_{-L}^{0} h(x)\,dx + \int_{0}^{L} h(x)\,dx$

Let $x = -u$
$\rightarrow dx = -du$

$= -\int_{L}^{0} h(-u)\,du + \int_{0}^{L} h(x)\,dx$ by u-substitution

$= \int_{0}^{L} h(-x)\,dx + \int_{0}^{L} h(x)\,dx$, switch order of int'n & rename the dummy var.

$= -\int_{0}^{L} h(x)\,dx + \int_{0}^{L} h(x)\,dx$ bec. h odd.

$= 0$ ✓

(c). f odd, g odd; show fg even:

$fg(-x) = f(-x)g(-x)$ by def of mult'n of fnc's

$= [-f(x)][-g(x)]$ bec f & g both odd

$= fg(x)$. So fg is even by def. ✓

(d) h even $\int_{-L}^{L} h(x)\,dx = \int_{-L}^{0} h(x)\,dx + \int_{0}^{L} h(x)\,dx$

Let $x = -u$
$\rightarrow dx = -du$

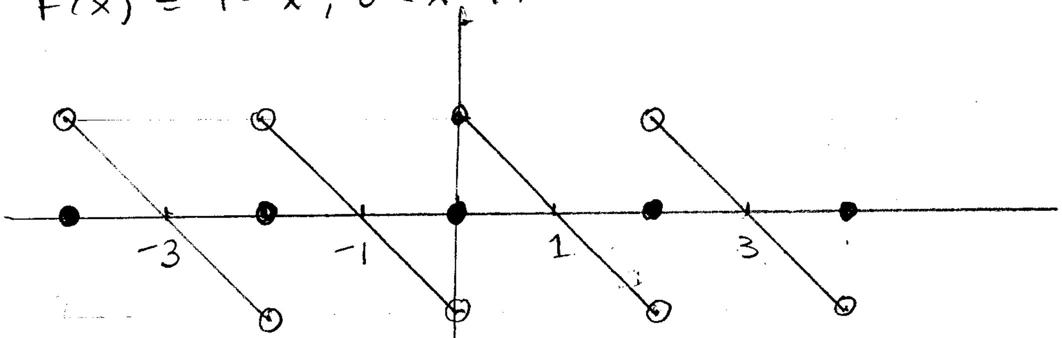
$= -\int_{L}^{0} h(-u)\,du + \int_{0}^{L} h(x)\,dx$ by u-substitution

$= \int_{0}^{L} h(-x)\,dx + \int_{0}^{L} h(x)\,dx$, switch order of int'n & rename the dummy var

$= \int_{0}^{L} h(x)\,dx + \int_{0}^{L} h(x)\,dx$ bec. h even.

$= 2\int_{0}^{L} h(x)\,dx$ ✓

(e) $f(x) = 1-x,\ 0<x<1.$



4 periods.

(f). $\underline{\sum_{n=1}^{\infty} b_n \sin n\pi x}$ $\quad b_n = \frac{2}{1}\int_0^1 \underbrace{(1-x)}_{u}\ \underbrace{\sin n\pi x\, dx}_{dv}.$

$$= 2\left\{ \frac{(1-x)\cos n\pi x}{n\pi}\Big|_0^1 - \int_0^1 \frac{\cos n\pi x}{n\pi}\, dx \right\} = {}^{+}\frac{2}{n\pi} = b_n$$

(g). Convergence is uniform on every closed interval that excludes points $x = 2n$, $n = 0, \pm1, \pm2, \ldots$

(*) See attached maple graph.

(h) f even, g even; show fg even:

$fg(-x) = f(-x)g(-x)$ by def of mult'n of fnc's

$= f(x)g(x)$ bec f & g both even.

$= fg(x)$. So fg is even by def. ✓

(i). $f(x)$ is even.

$\cos\frac{n\pi x}{L}$ is even, $n = 0, 1, 2, \ldots$

$f(x)\cos\frac{n\pi x}{L}$ is even by (1h).

$\Rightarrow a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx,$

$n = 0, 1, 2, 3, \ldots$ by (1d).

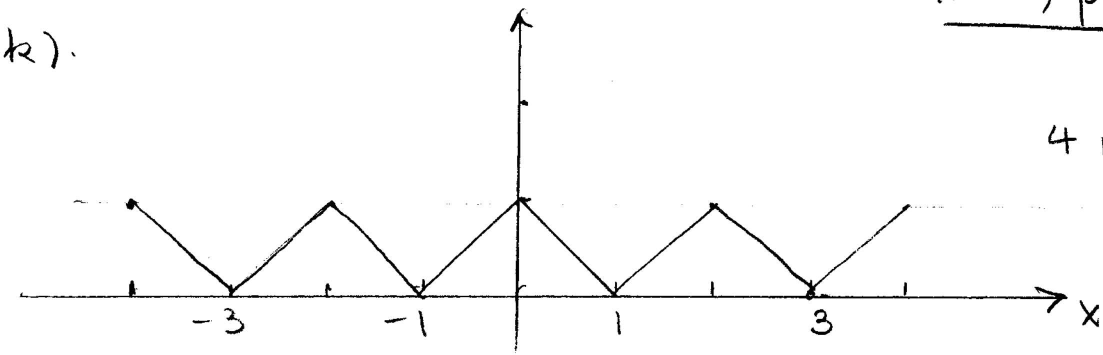
(j) $f(x)$ is even

$\sin\frac{n\pi x}{L}$ is odd, $n = 1, 2, 3, \ldots$

$f(x)\sin\frac{n\pi x}{L}$ is odd by (1a).

$\Rightarrow b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\, dx = 0$, $n = 1, 2, 3, \ldots$ by (1b).

(k).



4 periods.

(l). $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x = g(x)$.

$$\frac{a_0}{2} = \int_0^1 (1-x)\,dx = x - \frac{x^2}{2}\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2} = \frac{a_0}{2}$$

$$a_n = 2\int_0^1 \underbrace{(1-x)}_{u} \underbrace{\cos n\pi x\,dx}_{dv}$$

$$= 2\left\{ (1-x)\frac{\sin n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx \right\}$$

$$= -2\,\frac{\cos n\pi x}{n^2\pi^2}\Big|_0^1 = \frac{-2}{n^2\pi^2}\left[(-1)^n - 1\right] = a_n,$$

$n = 1, 2, 3, \ldots$

(m) convergence is uniform on $\mathbb{R}$.

For all $\varepsilon > 0$ there exists a corresponding $N^* > 0$ such that for all $N > N^*$

$$|S_N(x) - g(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

(*) see attached maple graph.

> #Sine series and cosine series

> $b := k \rightarrow \frac{2}{1} \cdot int((1-x)\cdot\sin(k\cdot\text{Pi}\cdot x), x = 0..1);$

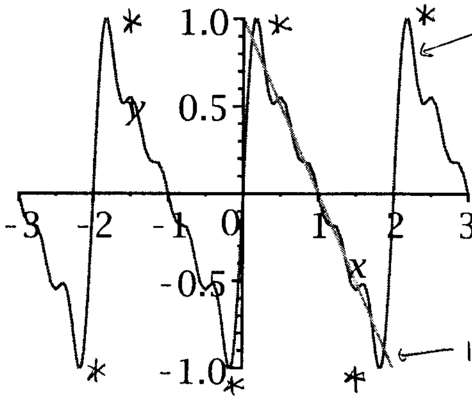
$$b := k \rightarrow 2 \left(\int_0^1 (1-x)\sin(k\pi x)\, dx \right) \qquad (1)$$

> $ss := n \rightarrow sum(b(k)\cdot\sin(k\cdot\text{Pi}\cdot x), k = 1..n);$

$$ss := n \rightarrow \sum_{k=1}^{n} b(k)\sin(k\pi x) \qquad (2)$$

> $plot(\{1-x, ss(5)\}, x = -3..3, y = -1..1);$

* overshoot = Gibbs' phenomenon



$S_5(x)$

$1-x = f(x)$

F.S.S.

> $a := k \rightarrow \frac{2}{1} \cdot int((1-x)\cdot\cos(k\cdot\text{Pi}\cdot x), x = 0..1);$

$$a := k \rightarrow 2 \left(\int_0^1 (1-x)\cos(k\pi x)\, dx \right) \qquad (3)$$

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> sc := n -> 1/2 + sum(a(k)*cos(k*Pi*x), k = 1 .. n);
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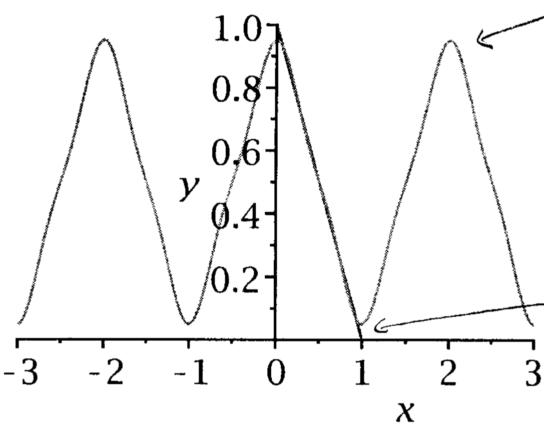
$$sc := n \rightarrow \frac{1}{2} + \sum_{k=1}^{n} a(k)\cos(k\pi x) \qquad (4)$$

> $plot(\{1-x, sc(4)\}, x = -3..3, y = 0..1);$

no Gibbs' phenomenon (no points of discontinuity)

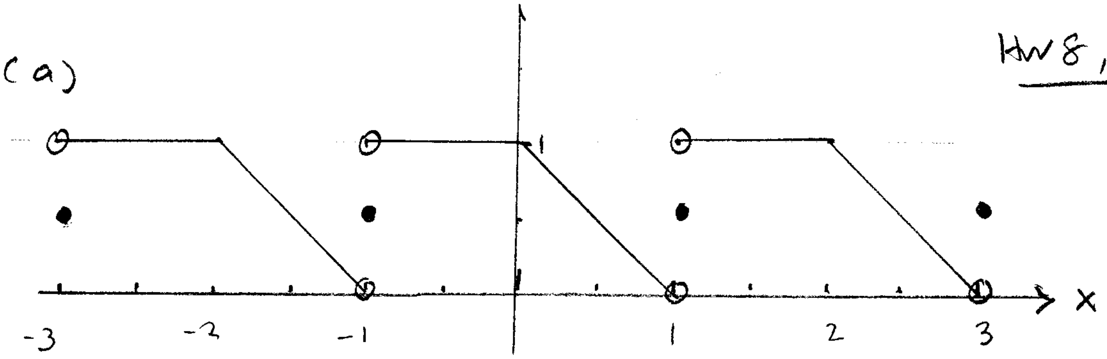


$S_5(x)$

$1-x = f(x)$

F.C.S.

(2) (a)



(b) $\boxed{\dfrac{a_0}{2} + \sum_{n=1}^{\infty} A_n \sin(n\pi x + \phi_n)}$ [(c) see attached]

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \tan\phi_n = \frac{a_n}{b_n}$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \underbrace{\int_0^1 (1-x)\,dx}_{\text{see (1e)}} = 1 + \frac{1}{2} = \boxed{\frac{3}{2} = a_0}$$

$$a_n = \frac{1}{1}\left\{ \int_{-1}^{0} \cos n\pi x\,dx + \underbrace{\int_0^1 (1-x)\cos n\pi x\,dx}_{\text{see (1e)}} \right\} \Rightarrow$$

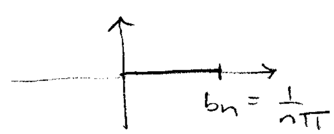
$$a_n = \frac{1-(-1)^n}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{1}\left\{ \int_{-1}^{0} \sin n\pi x\,dx + \underbrace{\int_0^1 (1-x)\sin n\pi x\,dx}_{\text{see (1f)}} \right\}$$

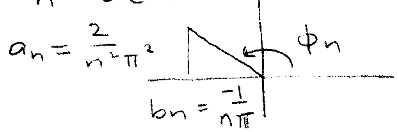
$$= \left.\frac{-\cos n\pi x}{n\pi}\right|_{-1}^{0} + \frac{1}{n\pi} = -\frac{1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{1}{n\pi} = \frac{(-1)^n}{n\pi}$$

$$a_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{2}{n^2\pi^2}, & n \text{ odd} \end{cases} \qquad b_n = \begin{cases} \dfrac{1}{n\pi}, & n \text{ even} \\ \dfrac{-1}{n\pi}, & n \text{ odd} \end{cases}$$

n even



n odd



$$\boxed{\phi_n = \begin{cases} 0, & n = 2, 4, 6, \ldots \\ \tan^{-1}\left(\dfrac{2}{n^2\pi^2} \cdot \dfrac{n\pi}{(-1)}\right) + \pi, & n = 1, 3, 5, \ldots \end{cases}}$$

$$\boxed{A_n = \sqrt{\left(\frac{1-(-1)^n}{n^2\pi^2}\right)^2 + \left(\frac{1}{n\pi}\right)^2} = \begin{cases} \dfrac{1}{n\pi}, & n = 2, 4, 6, \ldots \\ \sqrt{4 + n^2\pi^2}\,/(n^2\pi^2), & n = 1, 3, 5, \ldots \end{cases}}$$

> $A := n \rightarrow \mathrm{sqrt}\left(\frac{(1-(-1)^n)^2}{n^4\pi^4} + \frac{1}{n^2\pi^2}\right);$

$$A := n \rightarrow \sqrt{\frac{(1-(-1)^n)^2}{n^4\pi^4} + \frac{1}{n^2\pi^2}} \tag{5}$$

> $A(1);\ \mathit{evalf}(A(1));$

$$\sqrt{\frac{4}{\pi^4} + \frac{1}{\pi^2}}$$
$$0.3773395190 \tag{6}$$

> $A(2);\ \mathit{evalf}(A(2));$

$$\frac{1}{2\pi}$$
$$0.1591549430 \tag{7}$$

> $A(3);\ \mathit{evalf}(A(3));$

$$\frac{1}{9}\sqrt{\frac{4}{\pi^4} + \frac{9}{\pi^2}}$$
$$0.1084659918 \tag{8}$$

> $A(4);\ \mathit{evalf}(A(4));$

$$\frac{1}{4\pi}$$
$$0.07957747152 \tag{9}$$

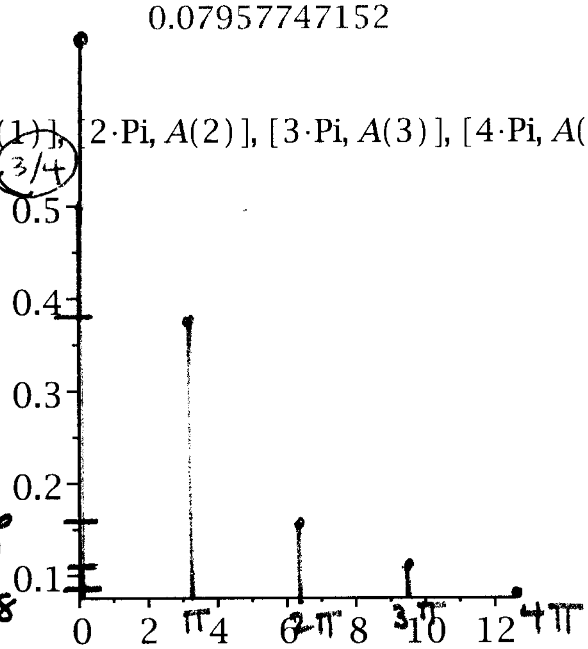
> ?*plot*

3/4

> $\mathit{plot}\left(\left[\left[0, \cancel{\frac{1}{2}}\right], [\mathrm{Pi}, A(1)], [2\cdot\mathrm{Pi}, A(2)], [3\cdot\mathrm{Pi}, A(3)], [4\cdot\mathrm{Pi}, A(4)]\right], \mathit{style} = \mathit{point}\right);$

$\frac{a_0}{2} = \cancel{\frac{1}{2}}$ 3/4

$A_1 \approx 0.38$



$A_2 \approx 0.16$
$A_3 \approx 0.11$
$A_4 \approx 0.08$

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