Hw 7 - A sol$^{\underline{n}}$

(1) $\dfrac{x^2 y'' + xy' + x^2\lambda y = 0}{x}$, $0<x<1$.

(a)

$\rightarrow$ $(xy')' + x\lambda y = 0$

Form $(p(x)y')' + [-q(x) + \lambda r(x)]y = 0$,

where $p(x) = x$, $q(x) = 0$, $r(x) = x$.

(A). p, p', q, r are continuous on $[0,1]$.

(B) $p, r > 0$ on $(0,1]$ but not at $x=0$.

(C) The BC $y(1)=0$ is separated & the BC at 0 is an appropriate boundedness condition.

Problem is a singular S-L prob.

(b). Solution to the ODE:

(I) $\lambda = 0$: $x^2 y'' + xy' = 0$ $\leftarrow$ Euler eq.

$y = x^m \rightarrow m(\cancel{m-1}) + \cancel{m} = 0 \rightarrow m_1 = m_2 = 0$.

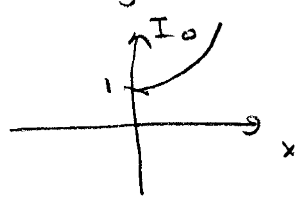
$y = c_1 x^0 + c_2 x^0 \ln x = c_1 + c_2 \ln x$.

$\lim_{x\to 0^+} |y| < \infty \rightarrow c_2 = 0 \Rightarrow y = c_1$

$y(1) = c_1 = 0 \rightarrow y(x) \equiv 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

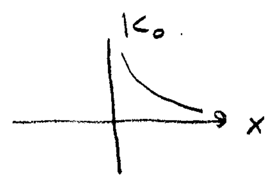
(II) $\lambda < 0$. Let $\lambda = -\alpha^2$, $\alpha > 0$.

$x^2 y'' + xy' - x^2\alpha^2 y = 0$



$\Rightarrow y = c_1 I_0(\alpha x) + c_2 K_0(\alpha x)$

$\lim_{x\to 0^+} |y| < \infty \Rightarrow c_2 = 0$



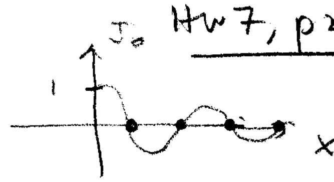
$\Rightarrow y = c_1 I_0(\alpha x)$

$y(1) = c_1 I_0(\alpha) = 0$

I_0 has no roots $\Rightarrow c_1 = 0 \Rightarrow y(x) \equiv 0 \Rightarrow$

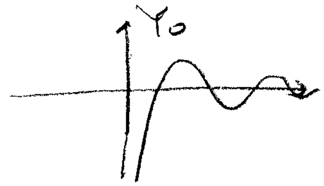
There are no negative eigenvalues λ.

(III) $\lambda > 0$. Let $\lambda = \beta^2$, $\beta > 0$.



$$x^2 y'' + x y' + x^2 \beta^2 y = 0$$

$$\Rightarrow y = c_1 J_0(\beta x) + c_2 Y_0(\beta x).$$



$$\lim_{x \to 0^+} |y| < \infty \Rightarrow c_2 = 0.$$

$$\Rightarrow y = c_1 J_0(\beta x)$$

$y(1) = c_1 J_0(\beta) = 0 \to \beta = b_n$, n^{th} root of $J_0(x)$.

$$\Rightarrow y = c_1 J_0(b_n x)$$

$$\boxed{\begin{array}{l}\text{Eigenvalues are } \lambda_n = b_n^2. \\ \text{Corresponding eigenfunctions are } y_n = J_0(b_n x) \\ \text{where } b_n \text{ is the } n^{\text{th}} \text{ root of } J_0(x),\ n = 1, 2, 3, \ldots\end{array}}$$

(c). See the sketch of $J_0(x)$ above.

(i) The eigenvalues $\lambda_n = b_n$ are real

(ii) There are infinitely many distinct $\lambda_n = b_n$

(iii) $b_1 < b_2 < b_3 < b_4 < \cdots$

(iv) $\lim_{n \to \infty} b_n = \infty$

(d). $\langle y_n(x), y_m(x) \rangle = \int_0^1 y_n(x)\, y_m(x)\, x\, dx$

(e). Show $\int_0^1 J_0(b_n x) J_0(b_m x)\, x\, dx = 0$:

Let $J_0(b_n x) = u(x)$, $J_0(b_m x) = v(x)$ for convenience.

By definition of u & v,

$$x^2 u'' + x u' + b_n^2 x^2 u = 0$$

$$x^2 v'' + x v' + b_m^2 x^2 v = 0$$

or in S-L form:

$$\left.\begin{array}{ll}(x u')' + b_n^2 x u = 0 & (1) \\ (x v')' + b_m^2 x v = 0. & (2)\end{array}\right\}$$

mult (1) by v & (2) by u.
Subtract.
Integrate from 0 to 1:

$$\int_0^1 (x u')' v\, dx - \int_0^1 (x v')' u\, dx = \int_0^1 (b_m^2 - b_n^2)\, u v x\, dx.$$

Int by parts $\Rightarrow$ $v(x u')\Big|_0^1 - \cancel{\int_0^1 x u' v'\, dx} +$

$$- (x v') u \Big|_0^1 + \cancel{\int_0^1 x v' u'\, dx} = (b_m^2 - b_n^2) \int_0^1 u v x\, dx.$$

$v(1)\, x\, u'(1) - 0 - x v'(1) u(1) + 0$

$= (b_m^2 - b_n^2) \int_0^1 u v x \, dx.$

But $v(1) = J_0(b_m) = 0$ & $u(1) = J_0(b_n) = 0$ by def of b_m, b_n.

So $0 = (b_m^2 - b_n^2) \int_0^1 u v x \, dx.$

$m \neq n \Rightarrow \int_0^1 J_0(b_n x) J_0(b_m x)\, x \, dx = 0$ as desired.

(2). Find the F.S. of $f(x) = \begin{cases} 2, & -2 \le x < 0 \\ x, & 0 \le x \le 2. \end{cases}$
$L = 2.$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx. = \frac{1}{2}\left\{ \int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \right.$$

$$\left. + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx \right\}.$$

$$= \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{1}{2}\left\{\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx\right\}$$

if $n \neq 0$.

$$= \frac{1}{(n\pi)^2}2\cos\frac{n\pi x}{2}\Big|_0^2 = \frac{2}{(n\pi)^2}\{\cos(n\pi) - 1\}$$

$$\underline{a_n = \frac{2}{(n\pi)^2}\{(-1)^n - 1\}, \quad n = 1, 2, 3, \ldots}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left\{\int_{-2}^{0} 2\,dx + \int_0^2 x\,dx\right\} =$$

$$= \frac{1}{2}\left\{2x\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^2\right\} = \frac{1}{2}\{+4+2\} = \underline{3 = a_0}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left\{\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \right.$$

$$\left. + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right\}$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{1}{2}\left\{-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_0^2 + \int_0^2 \frac{+2}{n\pi}\cos\frac{n\pi x}{2}\,dx\right\}$$

if $n \neq 0$ ✓

$$= -\frac{2}{n\pi}(1 - \cancel{\cos(n\pi)}) + \left(-\cancel{\frac{2}{n\pi}\cos n\pi}\right) +$$

$$\frac{1}{(n\pi)^2}2\sin\frac{n\pi x}{2}\Big|_0^2 = \underline{-\frac{2}{n\pi} = b_n, \quad n = 1, 2, 3, \ldots}$$

$$\boxed{F.S. = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{(n\pi)^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]}$$

$S_0 = \frac{3}{2}$, $S_1 = \frac{3}{2} - \frac{4}{\pi^2}\cos\frac{\pi x}{2} - \frac{2}{\pi}\sin\frac{\pi x}{2}$

$S_2 = \frac{3}{2} - \frac{4}{\pi^2}\cos\frac{\pi x}{2} - \frac{2}{\pi}\sin\frac{\pi x}{2} - \frac{1}{\pi}\sin\pi x, \ldots$