HW6 - A Sol$^{\underline{n}}$

AEM II
Gross
Sp2008

(1) $y'' + \lambda y = 0,\ -L < x < L,\ y(-L) = y(L),\ y'(-L) = y'(L)$

(a) $\lambda = 0$: $y'' = 0 \to y = c_1 x + c_2$

$$\left.\begin{aligned} y(-L) &= c_1(-L) + c_2 \\ y(L) &= c_1 L + c_2 \end{aligned}\right] \stackrel{=}{\Rightarrow} \begin{aligned} -c_1 L &= c_1 L \\ L \neq 0 &\Rightarrow c_1 = 0 \end{aligned}$$

$y(x) = c_2 \to y'(x) \equiv 0$

$$\left.\begin{aligned} y'(-L) &= 0 \\ y'(L) &= 0 \end{aligned}\right] \stackrel{=}{} \checkmark \quad \Rightarrow y(x) = c_2(1)$$

Eigenfunction corresponding to eigenvalue $\lambda = 0$ is $\boxed{y(x) \equiv 1.}$

(b) $\lambda < 0$: $y = e^{rx} \to r^2 + \lambda = 0 \to r = \pm\sqrt{-\lambda}$ real

$y = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

$y(-L) = y(L) \Rightarrow$

$c_1 e^{-\sqrt{-\lambda}L} + c_2 e^{+\sqrt{-\lambda}L} = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$

or $\quad c_1\left(e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}\right) = c_2\left(e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}\right)$

$\to c_1 = c_2$ bec $e^{-\sqrt{-\lambda}L} \neq e^{\sqrt{-\lambda}L}$ for $L \neq 0,\ \lambda < 0$, (*)

so $y = c_1\left(e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}\right)$

$y' = c_1\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}\right)$

$y'(-L) = y'(L) \Rightarrow$

$c_1\cancel{\sqrt{-\lambda}}\left(\cancel{e^{-\sqrt{-\lambda}L} - e^{\sqrt{-\lambda}L}}\right) = c_1\cancel{\sqrt{-\lambda}}\left(\cancel{e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}}\right)$ -1 (*)

($\sqrt{-\lambda} \neq 0$ for $\lambda < 0$). So $c_1 = -c_1 \Rightarrow c_1 = 0$

$\Rightarrow y(x) \equiv 0 \Rightarrow$ $\boxed{\text{There are no negative eigenvalues.}}$

(1c) $\lambda > 0$: $r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$$y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$$

$$y(-L) = y(L) \Rightarrow \cancel{c_1\cos\sqrt{\lambda}L} - c_2\sin\sqrt{\lambda}L = \cancel{c_1\cos\sqrt{\lambda}L} + c_2\sin\sqrt{\lambda}L$$

$$\Rightarrow \underbrace{c_2 = 0}_{(i)} \quad \text{or} \quad \underbrace{\sqrt{\lambda}L = n\pi}_{(ii)}, \; n = 1, 2, 3, \ldots$$

$\downarrow$

$$y = c_1 \cos\sqrt{\lambda}x$$

$$y' = -(c_1 \sin\sqrt{\lambda}x)\sqrt{\lambda}$$

$$y'(-L) = y'(L) \Rightarrow$$

$$+\cancel{c_1}\cancel{\sqrt{\lambda}}\sin\sqrt{\lambda}L = -\cancel{c_1}\cancel{\sqrt{\lambda}}\sin\sqrt{\lambda}L$$

($\sqrt{\lambda} \neq 0$ for $\lambda > 0$)

want $c_1 \neq 0$ to avoid $y(x) \equiv 0$

so (ii) must hold

So $\sqrt{\lambda} = \frac{n\pi}{L}$.

so $y(x) = c_1 \cos\frac{n\pi x}{L} + c_2 \sin\frac{n\pi x}{L}$.

So $y'(x) = -c_1\frac{n\pi}{L}\sin\frac{n\pi x}{L} + c_2\frac{n\pi}{L}\cos\frac{n\pi x}{L}$

$$y'(-L) = y'(L) \Rightarrow c_1\frac{n\pi}{L}\sin n\pi + \cancel{c_2\frac{n\pi}{L}\cos n\pi}$$

$$= -c_1\frac{n\pi}{L}\sin n\pi + \cancel{c_2\frac{n\pi}{L}\cos n\pi} \Rightarrow 0 = 0 \checkmark$$

So c_1 & c_2 remain arbitrary

$$y(x) = c_1\cos\frac{n\pi x}{L} + c_2\sin\frac{n\pi x}{L}$$

$$\boxed{\text{eigenvalues: } \lambda_n = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, \ldots}$$

two families of solutions

Corresponding eigenfunctions: $\boxed{y_{1n}(x) = \cos\frac{n\pi x}{L}, \quad y_{2n}(x) = \sin\frac{n\pi x}{L}}$.

(1d) (A). $\left\langle \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \right\rangle = \int_{-L}^{L} \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx =$

$n = 1, 2, 3, \ldots$
$m = 1, 2, 3, \ldots$

$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$ $\quad A = \frac{n\pi x}{L}$, $B = \frac{m\pi x}{L}$.

$$= \frac{1}{2}\int_{-L}^{L}\left[\sin\frac{(n+m)\pi x}{L} - \underbrace{\sin\frac{(n-m)\pi x}{L}}_{\text{This term absent if } n=m.}\right]dx = 0 \checkmark$$

because of evenness of cosine.

(B). $\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L} \rangle_{\substack{n=1,2,3,\dots;\ m=1,2,3,\dots \\ n\neq m}} = \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx =$

$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B \qquad A = \frac{n\pi x}{L},\ B = \frac{m\pi x}{L}$

$= \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(n+m)\pi x}{L} + \cos\frac{(n-m)\pi x}{L}\right]dx = 0$ ✓ because $\sin k\pi = 0$ for $k\in\mathbb{Z}$.

(C) $\langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \rangle_{\substack{n=1,2,3,\dots;\ m=1,2,3,\dots \\ n\neq m}} = 0$ ✓ similar to above & similar to class work.

(D). $\langle 1, \cos\frac{n\pi x}{L} \rangle_{n=1,2,3,\dots} = \int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} = 0-0$ ✓

(E) $\langle 1, \sin\frac{n\pi x}{L} \rangle_{n=1,2,3,\dots} = \int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_{-L}^{L} =$

$= -\frac{L}{n\pi}\left[\cos n\pi - \cos n\pi\right] = 0$ ✓

(2) $y'' + \lambda y = 0,\ 0 < x < 1,\ \alpha y(0) + y'(0) = 0,\ y(1) = 0$

(a) $\lambda > 0$: $y = e^{rx} \to r^2 = -\lambda \to r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x.$

$y' = -c_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x.$

$$\left.\begin{aligned} \alpha y(0) + y'(0) &= \alpha c_1 + c_2\sqrt{\lambda} = 0 \quad (i) \\ y(1) &= c_1 \cos\sqrt{\lambda} + c_2 \sin\sqrt{\lambda} = 0 \quad (ii) \end{aligned}\right\} \text{ Solve.}$$

(i) $\to$ $c_2 = -\dfrac{\alpha c_1}{\sqrt{\lambda}}$ sub into (ii) $\to$ $c_1\left[\cos\sqrt{\lambda} - \dfrac{\alpha}{\sqrt{\lambda}} \sin\sqrt{\lambda}\right] = 0$

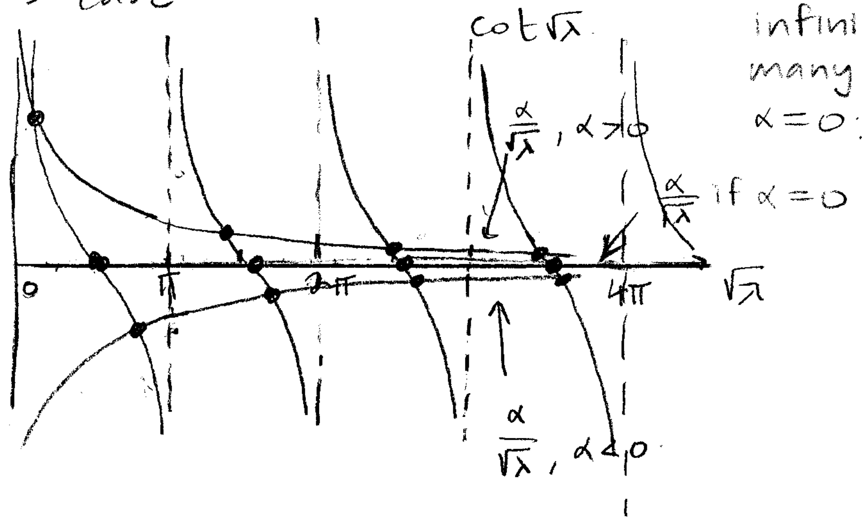
$\to$ $\cancel{\underbrace{c_1 = 0}}$ ~~(i) $\to$ $c_2 = 0$ bec $\sqrt{\lambda} > 0$~~ $\Downarrow$ ~~$y(x) \equiv 0$~~ or $\underbrace{\cos\sqrt{\lambda} - \dfrac{\alpha}{\sqrt{\lambda}} \sin\sqrt{\lambda} = 0}$

avoid this case

$\downarrow$

$\dfrac{\alpha}{\sqrt{\lambda}} \sin\sqrt{\lambda} = \cos\sqrt{\lambda}.$

$\dfrac{\alpha}{\sqrt{\lambda}} = \cot\sqrt{\lambda}.$ the graphs of the two functions intersect at infinitely many pts if $\alpha = 0$; $\alpha > 0$, $\alpha < 0$ ✓



② $\lambda = 0$: $y'' = 0 \to y = c_1 x + c_2$

$y'(x) = c_1.$

$$\left.\begin{aligned} \alpha y(0) + y'(0) &= \alpha c_2 + c_1 = 0 \quad (i) \\ y(1) &= c_1 + c_2 = 0 \quad (ii) \end{aligned}\right\} \text{ solve}$$

$\underbrace{(i) \to c_1 = -\alpha c_2 \quad (ii) \to c_1 = -c_2}$

$-\alpha c_2 = -c_2 \to \alpha = 1.$ Else $c_2 = 0$ & $c_1 = 0$.

(2c) cont'd

$\alpha = 1 \rightarrow y = c_1 x - c_1 = c_1(x-1)$.

If $\alpha = 1$, $\lambda = 0$ is an eigenvalue, & the corresponding eigenfunction is $y(x) = x - 1$.

If $\alpha \neq 1$, $y(x) \equiv 0 \Rightarrow \lambda = 0$ is not an eigenvalue.

(d) $\lambda < 0$: $y = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

$y' = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\,x} - c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\,x}$

$\alpha y(0) + y'(0) = \alpha(c_1 + c_2) + c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = 0$

$\Rightarrow c_1(\alpha + \sqrt{-\lambda}) + c_2(\alpha - \sqrt{-\lambda}) = 0$ (i)

$y(1) = c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0$ (ii)

} solve

(ii) $\rightarrow c_2 = \dfrac{-c_1 e^{\sqrt{-\lambda}}}{e^{-\sqrt{-\lambda}}} = -c_1 e^{2\sqrt{-\lambda}}$ (*) ← sub into eq (i)

$\rightarrow c_1\left[(\alpha + \sqrt{-\lambda}) - e^{2\sqrt{-\lambda}}(\alpha - \sqrt{-\lambda})\right] = 0$

$\rightarrow$ $c_1 = 0$

(*) $\rightarrow c_2 = 0$

bec $e^{2\sqrt{-\lambda}} \neq 0$

$\Downarrow$

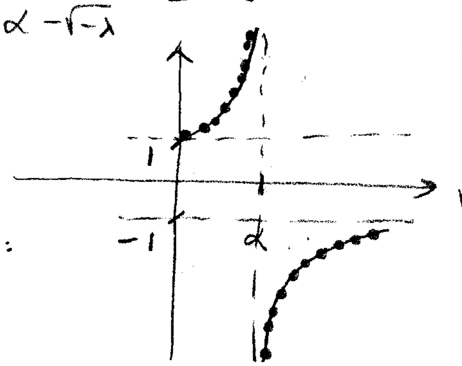
$y(x) \equiv 0$

avoid this case

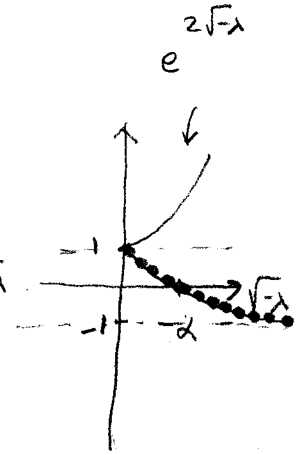
or $\dfrac{\alpha + \sqrt{-\lambda}}{\alpha - \sqrt{-\lambda}} = e^{2\sqrt{-\lambda}}$

rational fnc of $w = +\sqrt{-\lambda}$

Graph:

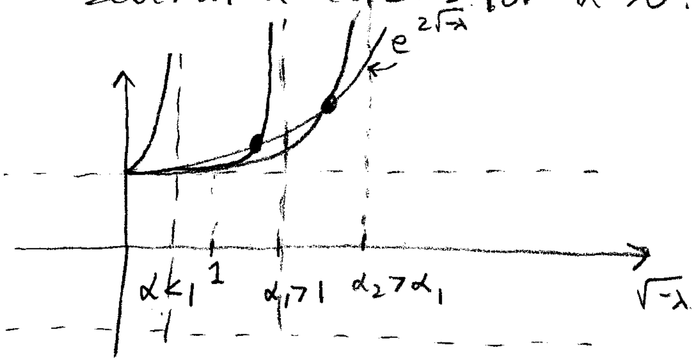


$\alpha > 0$



$\alpha < 0$: no sol'n for $\lambda \neq 0$.

several α cases for $\alpha > 0$:



Graphical evidence suggests that:

$\alpha \leq 1$ — no sol'n.

$\alpha > 1$ — one sol'n $\Rightarrow$ 1 neg e-val.

as α increases, intersection value $\sqrt{-\lambda}$ increases $\rightarrow -\lambda$ increases.

$\rightarrow \lambda$ decreases

Alternate way to write the sol'n in (2d):

(2d) $\lambda < 0 \to \quad y = c_1 \sinh(\sqrt{-\lambda}\,x) + c_2 \cosh(\sqrt{-\lambda}\,x)$

$$y' = c_1\sqrt{-\lambda}\cosh(\sqrt{-\lambda}\,x) + c_2\sqrt{-\lambda}\sinh(\sqrt{-\lambda}\,x)$$

$$\alpha y(0) + y'(0) = \alpha c_2 + c_1\sqrt{-\lambda} = 0 \to c_1 = -\frac{\alpha c_2}{\sqrt{-\lambda}} \quad (\dagger).$$

$$y(1) = c_1 \sinh\sqrt{-\lambda} + c_2\cosh\sqrt{-\lambda} = 0$$

$$c_2\left[-\frac{\alpha}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda} + \cosh\sqrt{-\lambda}\right] = 0$$

$\cancel{\to c_2 = 0}$ or $\quad -\frac{\alpha}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda} + \cosh\sqrt{-\lambda} = 0$

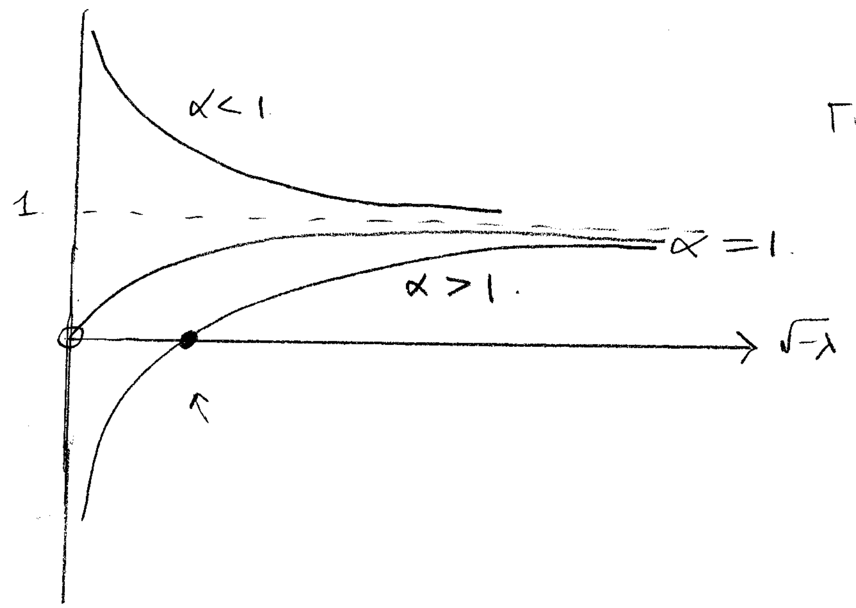
$\cancel{(\dagger) \to c_1 = 0}$ $\quad \Rightarrow \quad +\frac{\alpha}{\sqrt{-\lambda}} = +\coth\sqrt{-\lambda}$

$\cancel{\Downarrow}$

$\cancel{y(x) \equiv 0}$

avoid this case

$$\coth\sqrt{-\lambda} - \frac{\alpha}{\sqrt{-\lambda}}$$



Typical graphs.

Only for $\alpha > 1$ do we get a sol'n. (one negative eigenvalue λ).

(2b). For $\alpha < 1$,

$\lambda = 0$ is not an eigenvalue.
Part (c) showed $\lambda = 0$ is only an eigenvalue for $\alpha = 1$.

For $\alpha < 1$,

There are no negative eigenvalues.
Part (d) showed graphical evidence for this.

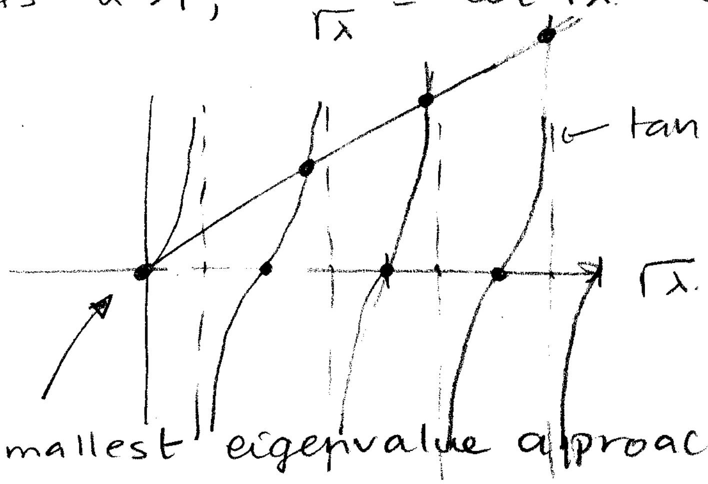
For $\alpha < 1$,

There are infinitely many positive eigenvalues. Part (a) showed infinitely many positive eigenvalues for all α.

Therefore, for $\alpha < 1$ all real eigenvalues are positive.

Part (a) showed positive eigenvalues satisfy $\frac{\alpha}{\sqrt{\lambda}} = \cot\sqrt{\lambda}$.

As $\alpha \to 1$, $\frac{1}{\sqrt{\lambda}} = \cot\sqrt{\lambda} \iff \sqrt{\lambda} = \tan\sqrt{\lambda}$.



smallest eigenvalue approaches 0.

(3a). $\boxed{A\cdot 1 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad 0<x<L}$, where.

$$A = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^L f(x)\,dx}{\int_0^L 1\cdot dx} = \boxed{\frac{1}{L}\int_0^L f(x)\,dx = A}$$

$$a_n = \frac{\langle f, \cos\frac{n\pi x}{L}\rangle}{\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\rangle} = \frac{\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}{\frac{1}{2}\int_0^L \left(1+\cos\frac{2n\pi x}{L}\right)dx}$$

$$\rightarrow \boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n=1,2,3,\ldots}$$

(3b) $\boxed{\sum_{n=1}^{\infty} c_n e^{3x}\sin\frac{n\pi x}{8}, \quad 0<x<8.}$

$$c_n = \frac{\langle f, e^{3x}\sin\frac{n\pi x}{8}\rangle}{\langle e^{3x}\sin\frac{n\pi x}{8}, e^{3x}\sin\frac{n\pi x}{8}\rangle} = \frac{\int_0^8 f(x)e^{3x}\sin\frac{n\pi x}{8}e^{-6x}\,dx}{\int_0^8 e^{6x}\sin^2\frac{n\pi x}{8}e^{-6x}\,dx}$$

$$= \frac{\int_0^8 f(x)e^{-3x}\sin\frac{n\pi x}{8}\,dx}{\frac{1}{2}\int_0^8 \left(1-\cos\frac{2n\pi x}{8}\right)dx} \Rightarrow \boxed{c_n = \frac{1}{4}\int_0^8 f(x)e^{-3x}\sin\frac{n\pi x}{8}\,dx}$$

(3c). $\boxed{\sum_{n=1}^{\infty} c_n x\sin\left(\frac{n\pi}{4}\ln x\right), \quad 1<x<e^4}$

$$c_n = \frac{\langle f, x\sin\left(\frac{n\pi}{4}\ln x\right)\rangle}{\langle x\sin\left(\frac{n\pi}{4}\ln x\right), x\sin\left(\frac{n\pi}{4}\ln x\right)\rangle} = \frac{\int_1^{e^4} f(x)\,x\sin\left(\frac{n\pi}{4}\ln x\right)x^{-3}\,dx}{\int_1^{e^4} x^2\sin^2\left(\frac{n\pi}{4}\ln x\right)x^{-3}\,dx}$$

$$\text{denom} = \frac{1}{2}\int_1^{e^4}\frac{1}{x}\left[1-\cos\left(\frac{2n\pi}{4}\ln x\right)\right]dx = \frac{1}{2}\ln|x|\Big|_1^{e^4}$$

$$= \frac{1}{2}\cdot 4 = 2$$

$$\rightarrow \boxed{c_n = \frac{1}{2}\int_1^{e^4} f(x)\frac{1}{x^2}\sin\left(\frac{n\pi}{4}\ln x\right)dx}$$