AEM II
Gross
Sp 2008

HW 5 - A soln

(1). $y'' + \gamma y = 0$

$y = e^{rx} \rightarrow r^2 + \gamma = 0 \rightarrow r = \pm\sqrt{-\gamma}$

(a). $\gamma = 1 \rightarrow r = \pm i \rightarrow y = c_1 \cos x + c_2 \sin x$.

$y(0) = c_1 = 0$; $y(1) = c_2 \sin(1) = 0 \rightarrow c_2 = 0 \rightarrow$

$\boxed{y(x) \equiv 0}$

(c). $\gamma = \pi^2 \rightarrow r = \pm \pi i \rightarrow y = c_1 \sin \pi x + c_2 \cos \pi x$.

$y(0) + y(1) = c_2 - c_2 = 0$ ✓

$y'(0) + y'(1) = c_1 \pi \cos(0) - c_2 \pi \sin(0) +$

$+ c_1 \pi \cos(\pi) - c_2 \pi \sin(\pi) = 0 \Leftrightarrow$

$c_1\pi - c_1\pi = 0$ ✓ $\boxed{y(x) = c_1 \sin \pi x,\ c_2 \cos \pi x}$

(b) $\gamma = \pi^2 \rightarrow y = c_1 \sin \pi x + c_2 \cos \pi x$

$y(0) = c_2 = 0$; $y(1) = c_1 \sin(\pi) = 0$ ✓

$\boxed{y(x) = c_1 \sin \pi x}$

(d). (a) separated, (b) separated, (c) neither

All (a)-(c) are homogeneous & linear.

(2). $\mu(x) P(x) y'' + \mu(x) Q(x) y' + \mu(x) R(x; \lambda) y = 0. \quad (*)$

$(\mu P y')' + \mu R y = (\mu P)' y' + (\mu P) y'' + \mu R y =$

$= (\mu' P + \mu P') y' + (\mu P) y'' + \mu R y \quad (**)$

want $\cancel{\mu P y''} + \mu Q y' + \cancel{\mu R y}$ from $(*)$

$= \cancel{\mu P y''} + (\mu' P + \mu P') y' + \cancel{\mu R y}$ from $(**)$

$\rightarrow \mu Q = \mu' P + \mu P' \rightarrow \underline{P \mu' = (Q - P')\mu}$ ✓

$\frac{d\mu}{\mu} = \left(\frac{Q - P'}{P}\right) dx \overset{\text{int.}}{\Rightarrow} \ln \mu = \int \frac{Q}{P} dx - \ln P = \int \frac{Q}{P} dx + \ln(P^{-1})$

$\Rightarrow \underline{\mu = \frac{1}{P} e^{\int \frac{Q}{P} dx}}$ ✓

(3) $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, $x > 0$

$\mu = \frac{1}{x^2}\exp\left(\int \frac{1}{x}\,dx\right) = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}$

$xy'' + y' + \frac{x^2 - \nu^2}{x} y = 0$

$(xy')' + (x + \frac{1}{x}\lambda)y = 0$, $\lambda = -\nu^2$

$p(x) = x$, $q(x) = -x$, $r(x) = \frac{1}{x}$

(4) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, $-1 < x < 1$

$\Rightarrow [(1-x^2)y']' + \underbrace{\alpha(\alpha+1)}^{\lambda} y = 0$ $(\mu(x) \equiv 1)$

$p(x) = 1 - x^2$, $q(x) = 0$, $r(x) \equiv 1$.

(5a) $y'' + \lambda y = 0$

(I) $\lambda < 0$: $y = c_1 e^{+\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$y' = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$

BC. $y'(0) = c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = 0 \rightarrow c_1 = c_2$

$y(x) = c_1[e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}]$

BC: $y'(L) = c_1\sqrt{-\lambda}\left[e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right] = 0$

If $c_1 \neq 0$, then $\cancel{\sqrt{-\lambda} = 0}$ or $\cancel{e^{\sqrt{-\lambda}L} = e^{-\sqrt{-\lambda}L}}$

bec. $\lambda < 0$. bec. $\sqrt{-\lambda}L \neq 0$.

So $c_1 = 0$.

So $y(x) \equiv 0$

So there are no negative eigenvalues.

(5a cont'd) II $\lambda = 0$: $y = c_1 + c_2 x$

$y' = c_2$

BC: $y'(0) = c_2 = 0$

$\rightarrow$ $y(x) = c_1$

$y'(x) = 0$

BC: $y'(L) = 0$ ✓

So $\lambda = 0$ is an eigenvalue, & the corresponding eigenfunction is $y(x) \equiv 1$.

III $\lambda > 0$: $y = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$

$y' = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\, x) - c_2 \sin(\sqrt{\lambda}\, x)\sqrt{\lambda}$

BC: $y'(0) = c_1 \sqrt{\lambda} = 0 \rightarrow c_1 = 0$. (bec $\lambda > 0$)

$y(x) = c_2 \cos(\sqrt{\lambda}\, x)$

$y'(x) = -c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x)$

BC: $y'(L) = -c_2 \sqrt{\lambda} \sin(\sqrt{\lambda}\, L) = 0$

If $c_2 \neq 0$ then $\sqrt{\lambda} L = n\pi$, $n = 1, 2, 3, \ldots$

So $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$, are eigenvalues & the corresponding eigenfunctions are

$y_n(x) = \cos\left(\frac{n\pi}{L} x\right)$.

(5b) $(e^{-6x}y')' + (1+\lambda)e^{-6x}y = 0$, $y(0) = y(8) = 0$

I) $\lambda < 8$) $y = c_1 e^{(3+\sqrt{8-\lambda})x} + c_2 e^{(3-\sqrt{8-\lambda})x}$

BC: $y(0) = c_1 + c_2 = 0 \rightarrow c_2 = -c_1$

$y(x) = c_1 \left[e^{(3+\sqrt{8-\lambda})x} - e^{(3-\sqrt{8-\lambda})x} \right]$.

$y(8) = c_1 \left[e^{(3+\sqrt{8-\lambda})8} - e^{(3-\sqrt{8-\lambda})8} \right] = 0$

If $c_1 \neq 0$, need $\cancel{3} + \sqrt{8-\lambda} = \cancel{3} - \sqrt{8-\lambda} \rightarrow \lambda = 8$

So $c_1 = 0$, so $y(x) \equiv 0$ $\rightarrow\!\!\!\!\leftarrow$.

So there are no eigenvalues < 8.

II) $\lambda = 8$: $y = c_1 e^{3x} + c_2 e^{3x} x$

BC: $y(0) = c_1 = 0 \rightarrow y(x) = c_2 e^{3x} x$

BC. $y(8) = c_2 e^{24}(8) = 0 \rightarrow c_2 = 0$.

$\rightarrow \quad y(x) \equiv 0$.

So $\lambda = 8$ is not an eigenvalue.

III) $\lambda > 8$: $y = e^{3x} \left[c_1 \cos(\sqrt{\lambda-8}\, x) + c_2 \sin(\sqrt{\lambda-8}\, x) \right]$

BC: $y(0) = c_1 = 0$

$\rightarrow \quad y(x) = c_2 e^{3x} \sin(\sqrt{\lambda-8}\, x)$.

BC. $y(8) = c_2 e^{24} \sin(\sqrt{\lambda-8}\; 8) = 0$

If $c_2 \neq 0$ then $8\sqrt{\lambda-8} = n\pi$, $n = 1, 2, 3, \ldots$

So $\lambda_n = \left(\frac{n\pi}{8}\right)^2 + 8$, $n = 1, 2, 3, \ldots$, are eigenvalues & the corresponding eigenfunctions are $y_n(x) = e^{3x} \sin\left(\frac{n\pi}{8} x\right)$.

(5e) $\left(\frac{1}{x}y'\right)' + (4+\lambda)x^{-3}y = 0, \quad y(1) = y(e^4) = 0$

(I) $\lambda < -3$: $y = c_1 x^{1+\sqrt{-3-\lambda}} + c_2 x^{1-\sqrt{-3-\lambda}}$

BC: $y(1) = c_1 + c_2 = 0 \to c_2 = -c_1$

$y(x) = c_1\left[x^{1+\sqrt{-3-\lambda}} - x^{1-\sqrt{-3-\lambda}}\right]$

BC: $y(e^4) = c_1\left[e^{4(1+\sqrt{-3-\lambda})} - e^{4(1-\sqrt{-3-\lambda})}\right] = 0$

$\to c_1 = 0$ or $\cancel{\sqrt{-3-\lambda} = -\sqrt{-3-\lambda}}$

impossible for $\lambda < -3$.

So $c_1 = 0$

So $y(x) \equiv 0$.

So there are no eigenvalues < -3

(II) $\lambda = -3$. $y = c_1 x + c_2 x \ln x$

BC: $y(1) = c_1 = 0 \to y(x) = c_2 x \ln x$

BC: $y(e^4) = c_2 e^4 4 = 0 \to c_2 = 0 \to$

$y(x) \equiv 0 \to \lambda = -3$ is not an eigenvalue

(III) $\lambda > -3$: $y = x\left[c_1 \cos(\sqrt{3+\lambda}\ln x) + c_2 \sin(\sqrt{3+\lambda}\ln x)\right]$

BC: $y(1) = c_1 = 0 \to y = c_2 x \sin(\sqrt{3+\lambda}\ln x)$

BC: $y(e^4) = c_2 e^4 \sin(\sqrt{3+\lambda}\, 4) = 0$

If $c_2 \neq 0$ then $4\sqrt{3+\lambda} = n\pi, \; n = 1, 2, 3, \ldots$

So $\lambda_n = \left(\frac{n\pi}{4}\right)^2 - 3, \; n = 1, 2, 3, \ldots$ are eigenvalues

The corresponding eigenfunctions are

$y_n(x) = x \sin\left(\frac{n\pi}{4} \ln x\right)$.