Hw 4-A sol'n

AEM II
Gross
S2008.

(1) $r^2R'' + rR' + r^2\beta^2 R = 0, \quad 0<r<a.$

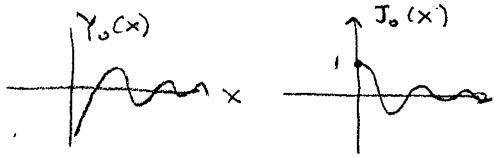
(a) $r = \dfrac{x}{\beta} \rightarrow x = \beta r.$ $\quad R(r) = R(r(x)) = \hat{R}(x).$

$$\frac{dR}{dr} = \frac{d\hat{R}}{dx}\beta \rightarrow \frac{d^2R}{dr^2} = \frac{d^2\hat{R}}{dx^2}\beta^2$$

$$\frac{x^2}{\cancel{\beta^2}}\cancel{\beta^2}\frac{d^2\hat{R}}{dx^2} + \frac{x}{\cancel{\beta}}\cancel{\beta}\frac{d\hat{R}}{dx} + \frac{x^2}{\cancel{\beta^2}}\cancel{\beta^2}\hat{R} = 0 \quad \leftarrow \text{Bessel's eq of order } 0.$$

(b) $\hat{R}(x) = c_1 J_0(x) + c_2 Y_0(x).$

$R(r) = c_1 J_0(\beta r) + c_2 Y_0(\beta r).$



$\lim_{r\to 0^+} |R(r)| < \infty \rightarrow c_2 = 0.$

$R(r) = c_1 J_0(\beta r).$

$R(a) = c_1 J_0(\beta a) = 0 \rightarrow c_1 \neq 0$ or $J_0(\beta a) = 0$

want nontrivial $R(r)$ — choose $\boxed{\beta = \dfrac{b_n}{a},}$

$\boxed{\text{where } b_n \text{ is a root of } J_0.}$ As such, there are infinitely many solutions: $n = 1, 2, 3, \ldots$:

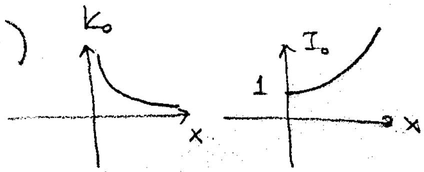
(c) $\boxed{R_n(r) = J_0\left(\dfrac{b_n}{a} r\right)}$

(2). $r^2R'' + rR' - r^2\alpha^2 R = 0, \quad 0<r<a.$

(a) $r = \dfrac{x}{\alpha} \rightarrow$ (like in (1a))

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} - x^2 R = 0 \quad \leftarrow \text{modified Bessel's eq of order } 0.$$

(b) $R(r) = c_1 I_0(\alpha r) + c_2 K_0(\alpha r)$



(c) $\lim_{r\to 0^+} |R(r)| < \infty \rightarrow c_2 = 0$

$R(r) = c_1 I_0(\alpha r).$

$R(a) = c_1 \underbrace{I_0(\alpha a)}_{} = 0$

nonzero for all α because I_0 has no roots.

$\rightarrow c_1 = 0 \rightarrow \boxed{R(r) \equiv 0}$

(3) $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2-1)^n$

$P_0(x) = (x^2-1)^0 = 1$ ✓

$P_1(x) = \dfrac{1}{2}\dfrac{d}{dx}(x^2-1) = \dfrac{1}{2}\,2x = x$ ✓

$P_2(x) = \dfrac{1}{4\cdot 2}\dfrac{d^2}{dx^2}(x^2-1)^2 = \dfrac{1}{8}\dfrac{d}{dx}\left(2(x^2-1)(2x)\right)$

$= \dfrac{4}{8}\dfrac{d}{dx}(x^3-x) = \dfrac{1}{2}(3x^2-1)$ ✓

$P_3(x) = \dfrac{1}{8\cdot 6}\dfrac{d^3}{dx^3}(x^2-1)^3 = \dfrac{1}{8\cdot 6}\dfrac{d^2}{dx^2}\left[3(x^2-1)^2(2x)\right] =$

$= \dfrac{1}{8}\dfrac{d^2}{dx^2}\left[x^5-2x^3+x\right] = \dfrac{1}{8}\dfrac{d}{dx}\left[5x^4-6x^2+1\right] =$

$= \dfrac{1}{8}\left[20x^3-12x\right] = \dfrac{5}{2}x^3-\dfrac{3}{2}x$ ✓

$P_4(x) = \dfrac{1}{16\cdot 24}\dfrac{d^4}{dx^4}\left[x^2-1\right]^4 = \dfrac{1}{\cancel{16}_{2}\cdot 24}\dfrac{d^3}{dx^3}\left[\cancel{4}(x^2-1)^3(\cancel{2}x)\right]$

$= \dfrac{1}{48}\dfrac{d^2}{dx^2}\left[3(x^2-1)^2(2x)(x) + (x^2-1)^3\right]$

$= \dfrac{1}{48}\dfrac{d^2}{dx^2}\left[(x^2-1)^2\left[\overset{7x^2}{\cancel{6x^2+x^2}}-1\right]\right]$

$= \dfrac{1}{48}\dfrac{d^2}{dx^2}\left[7x^6-x^4-14x^4+2x^2+7x^2-1\right]$

$= \dfrac{1}{48}\dfrac{d^2}{dx^2}\left[7x^6-15x^4+9x^2-1\right]$

$= \dfrac{1}{48}\dfrac{d}{dx}\left[42x^5-60x^3+18x\right]$

$= \dfrac{1}{48}\left[210x^4-180x^2+18\right]$

$= \dfrac{1}{8}\left[35x^4-30x^2+3\right]$ ✓

(4) (a) $\dfrac{-2x}{1-x^2}$, $\dfrac{\alpha(\alpha+1)}{1-x^2}$ are not analytic at $x=1$,

$(x-x_0)\dfrac{Q}{P} = (x-1)\dfrac{(-2x)}{1-x^2} = \dfrac{(+2x)}{(1+x)}$

$(x-x_0)^2\dfrac{R}{P} = (x-1)^2\dfrac{\alpha(\alpha+1)}{(1+x)(1-x)} = -\dfrac{\alpha(\alpha+1)(x-1)}{(1+x)}$

but these are analytic at $x=1$.

(4). $(1-x^2)\,y''(x) - 2x\,y'(x) + \alpha(\alpha+1)\cdot y(x) = 0$

$x = t+1. \qquad t = x-1.$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{d^2y}{dt^2}.$$

$$1-x^2 = 1-(t+1)^2 = 1-[t^2+2t+1] = -(t^2+2t) =$$

$$= -t(t+2).$$

(b). $-t(t+2)\,y''(t) - 2(t+1)\,y'(t) + \alpha(\alpha+1)\,y(t) = 0$

Note:

$t=0$ is a regular singular point.

$\dfrac{-2(t+1)}{-t(t+2)}$ & $\dfrac{\alpha(\alpha+1)}{-t(t+2)}$ are not diff'able at $t=0$, but

times t & times t^2 are analytic at $t=0$.

(c). Determine the indicial eq. & its roots for the point $x=1 \to t=0$:

$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}.$ sub into eq:

$n-1=m$
$n=m+1.$

$$\sum_{n=0}^{\infty} -\cancel{t^2}(n+r)(n+r-1)a_n t^{n+r-\cancel{2}} + \sum_{m=-1}^{\infty} -2\cancel{t}(m+1+r)(m+1+r-1)a_{m+1} t^{m+r-\cancel{2}} +$$

$$\sum_{n=0}^{\infty} -2\cancel{t}(n+r)a_n t^{n+r-\cancel{1}} + \sum_{m=-1}^{\infty} -2(m+1+r)a_{m+1} t^{m+r} + \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n t^{n+r}$$

$n-1=m$
$n=m+1$

$= 0.$

indicial eq: $m=-1$: $+\cancel{2}\,r(r-1)\cancel{a_0} + \cancel{2}\,r\,\cancel{a_0} = 0 \qquad a_0 \neq 0.$

$$r^2 = 0 \to \boxed{r_1 = r_2 = 0} = r.$$

(d).

$r=0$, index ≥ 0 : $-m(m-1)a_m - 2(m+1)m\,a_{m+1} +$

$- 2m\,a_m - 2(m+1)a_{m+1} + \alpha(\alpha+1)a_m = 0$

$$a_{m+1} = \frac{-m(m-1) - 2m + \alpha(\alpha+1)}{+2(m+1)[m+1]}\,a_m$$

$$= \frac{-m^2 - m + \alpha(\alpha+1)}{2(m+1)^2}\,a_m.$$

$m=0$: $a_1 = \dfrac{\alpha(\alpha+1)}{2}\,a_0$

$m=1$: $a_2 = \dfrac{-2+\alpha(\alpha+1)}{8}\,a_1 = \dfrac{[-2+\alpha(\alpha+1)]\,\alpha(\alpha+1)}{8\cdot 2}\,a_0.$

$y(t) = a_0 + a_1 t + a_2 t^2 + \cdots$; $t = x-1$; factor out $a_0 \rightarrow$.

$$y(x) = 1 + \frac{\alpha(\alpha+1)}{2}(x-1) + \frac{[-2+\alpha(\alpha+1)]\,\alpha(\alpha+1)}{16}(x-1)^2 + \cdots$$

$$+ (-1)^{n+1}\frac{\alpha(\alpha+1)[1\cdot 2 - \alpha(\alpha+1)]\cdots[n(n-1)-\alpha(\alpha+1)]}{2^n (n!)^2}\cdot(x-1)^n + \cdots.$$

✓