# HW 3 - A Sol'n

AEM II
Gross
S2008

(1a) $xy'' + (1-x)y' + y = 0, \quad x > 0$

(i) $\frac{Q}{P} = \frac{1-x}{x}, \quad \frac{R}{P} = \frac{1}{x}$ ← These are not analytic at zero, but

$x\frac{Q}{P} = 1-x, \quad x^2\frac{R}{P} = x$ ← these are analytic at zero

So $x=0$ is a regular singular point

(ii) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$-xy' = \sum_{n=0}^{\infty} -a_n(n+r)x^{n+r}$

$xy'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1}$

$m = n-1$

$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$

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$$\sum_{m=0}^{\infty} a_m x^{m+r} + \sum_{m=0}^{\infty} -a_m(m+r)x^{m+r} +$$

$$\sum_{m=-1}^{\infty} a_{m+1}(m+1+r)(m+r)x^{m+r} + \sum_{m=-1}^{\infty} a_{m+1}(m+r+1)x^{m+r} = 0$$

$m=-1$: $a_0[r(r-\cancel{1}) + \cancel{r}] = 0 \rightarrow r_1 = r_2 = 0$

($r^2 = 0$ is the indicial eq)

(iii) $m \geq 0$: $a_{m+1}[(m+1+r)(m+r) + (m+r+1)] +$

$a_m[1-(m+r)] = 0.$

$$a_{m+1} = \frac{-1+m+r}{(m+r+1)^2}\, a_m$$ ← recurrence relation

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(iv). $y_1 = \sum_{n=0}^{\infty} a_n x^{n+0}$

$$y_2 = \ln x \, y_1 + \sum_{n=1}^{\infty} b_n x^{n+0}$$

$y_1$: recurrence relation w/ $r=0$:

$m \geq 0$ $\quad a_{m+1} = \frac{m-1}{(m+1)^2} a_m. \quad m=0 \to a_1 = \frac{-1}{1} a_0.$

$m=1 \quad a_2 = 0 \to a_m = 0, \ m \geq 1. \Rightarrow y_1 = a_0 - a_0 x$

use $\boxed{y_1 = 1-x,}$ sub $y_2$ into the eq. →

$$y_2 = \underline{\underline{\ln x}}\, y_1 + \sum_{n=1}^{\infty} b_n x^n$$

$$-x\, y_2' = -\cancel{\tfrac{1}{x}}\, y_1 + -x \ln x\, y_1' + \sum_{n=1}^{\infty} -b_n n x^{\cancel{n-1}}$$

$m = n-1$

$$y_2' = \frac{1}{x} y_1 + \ln x\, y_1' + \sum_{n=1}^{\infty} b_n n x^{n-1}$$

$$x\, y_2'' = -\frac{1}{x^{\cancel{2}}} y_1 + \cancel{\tfrac{1}{x}} y_1' + \cancel{\tfrac{1}{x}} y_1' + x \ln x\, y_1'' + \sum_{n=2}^{\infty} b_n n(n-1) x^{\cancel{n-2}}$$

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$$\ln x \, \underbrace{[\, x y_1'' + y_1(\cancel{1-x}) + y_1 \,]}_{=0} - y_1 + 2y_1' + 4 \text{ series} = 0$$

$-y_1 = x-1, \quad 2y_1' = 2(-1).$

$$x - 3 + \sum_{m=1}^{\infty} b_m x^m + \sum_{m=1}^{\infty} -b_m m x^m + \sum_{m=0}^{\infty} b_{m+1}(m+1) x^m + \sum_{m=1}^{\infty} b_{m+1}(m+1) m x^m = 0.$$

equate like terms:

equate constant terms: $-3 + b_1 = 0 \to \underline{\underline{b_1 = 3}}$

equate $x^1$ terms: $1 + \cancel{b_1} - \cancel{b_1} + b_2 2 + b_2 2 = 0 \to \underline{\underline{b_2 = -\frac{1}{4}}}.$

equate $x^m$ terms $m \geq 2$:

$$b_m(1-m) + b_{m+1}\left((m+1)+(m+1)m\right) = 0.$$

$$b_{m+1} = \frac{m-1}{(m+1)^2} b_m.$$

$m=2$: $\quad b_3 = \frac{1}{9} b_2 = \frac{-1}{9}\left(\frac{1}{4}\right) = \frac{-1}{36} = b_3$

$m=3$: $\quad b_4 = \frac{2}{16} b_3 = \frac{1}{8}\left(-\frac{1}{36}\right) = -\frac{1}{288} = b_4$.

$$\boxed{y_2 = (\ln x)(1-x) + 3x - \frac{1}{4}x^2 - \frac{1}{36}x^3 - \frac{1}{288}x^4 + \dots}$$

Note $y = c_1 y_1 + c_2 y_2$

$$= c_1[1-x] + c_2\left[\ln x - x\ln x + 3x - \frac{1}{4}x^2 - \frac{1}{36}x^3 - \frac{1}{288}x^4 + \dots\right]$$

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addendum to problem (1b):

Note $y_1 = x^4 \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \dots)$ ← convergent geometric series for $|x|<1$. (Here we took $0<x<1$.)

$$= x^4 \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= x^4 (-1)(1-x)^{-2}(-1)$$

$$= \frac{x^4}{(1-x)^2}.$$

(equivalently, $y_1 = x^4 \frac{d}{dx}(x + x^2 + x^3 + x^4 + \dots)$

$$= x^4 \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{x^4}{(1-x)^2}$$ ).

(1b) $x(x-1)y'' + 3y' - 2y = 0, \quad x > 0.$

(i) $\dfrac{Q}{P} = \dfrac{3}{x(x-1)}, \quad \dfrac{R}{P} = \dfrac{-2}{x(x-1)}$ ← these are not analytic at $x=0$, but these are analytic at zero. Note $x=1$ is a singular point, so look for $y$ with $0<x<1$

$x\dfrac{Q}{P} = \dfrac{3}{x-1}, \quad x^2 \dfrac{R}{P} = -\dfrac{2x}{x-1}$

(ii) $-2y = \displaystyle\sum_{n=0}^{\infty} -2a_n x^{n+r}$

$3y' = \displaystyle\sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r-1}$ ← $m = n-1$

$x^2 y'' = \displaystyle\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r}$

$-xy'' = \displaystyle\sum_{n=0}^{\infty} -a_n (n+r)(n+r-1) x^{n+r-1}$

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$$\sum_{m=-1}^{\infty} -a_{m+1}(m+1+r)(m+r)x^{m+r} + \sum_{m=-1}^{\infty} 3a_{m+1}(m+1+r)x^{m+r}$$
$$+ \sum_{m=0}^{\infty} a_m (m+r)(m+r-1)x^{m+r} + \sum_{m=0}^{\infty} -2a_m x^{m+r} = 0.$$

$m=-1$: $a_0[-(r)(r-1) + 3r] = 0 \rightarrow -r(r-4) = 0$

indicial eq $\Rightarrow r = 0, 4.$

(iii) $m \geq 0$ $\quad a_{m+1}\left(-(m+1+r)(m+r) + 3(m+1+r)\right)$
$\quad + a_m\left((m+r)(m+r-1) - 2\right) = 0.$

$$a_{m+1} = \frac{-(m+r)(m+r-1) + 2}{(m+1+r)(-m-r+3)} a_m.$$ recurrence relation

let $r=4$: $\quad a_{m+1} = \dfrac{-(4+m)(3+m) + 2}{(m+5)(-m-1)} a_m$

$$= \frac{+[12 + 7m + m^2 - 2]}{+(m+5)(m+1)} a_m = \frac{\cancel{(m+5)}(m+2)}{\cancel{(m+5)}(m+1)} a_m$$

(1b)

(iv) $m=0: \quad a_1 = \frac{2}{1} a_0$

$m=1: \quad a_2 = \frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{1} a_0$

$m=2: \quad a_3 = \frac{4}{\cancel{3}} \cdot \cancel{3} a_0$

$m=3: \quad a_4 = \frac{5}{\cancel{4}} \cdot \cancel{4} a_0$

$$a_m = (m+1) a_0$$

$$\boxed{y_1 = \sum_{n=0}^{\infty} (n+1) x^{n+4}} = x^4 \left[1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\right]$$

Sub $y_2$ into the eq $\rightarrow$

$$-2y_2 = -2k \ln x \, y_1 + \sum_{n=0}^{\infty} -2 b_n x^{n+0}$$

$m = n-1$

$$3y_2' = k\frac{3}{x} y_1 + 3k \ln x \, y_1' + \sum_{n=1}^{\infty} 3 b_n x^{n-1} n$$

$$x^2 y_2'' = -k \frac{1}{\cancel{x^2}} y_1 + k\frac{x}{\cancel{x}} y_1' + k \frac{x}{\cancel{x}} y_1' + k x^2 \ln x \, y_1'' + \sum_{n=2}^{\infty} b_n n(n-1) x^{n}$$

$$-x y_2'' = +k \frac{1}{\cancel{x^2}} y_1 - k\frac{1}{\cancel{x}} y_1' - k \frac{1}{\cancel{x}} y_1' - x k \ln x \, y_1'' + \sum_{n=2}^{\infty} -b_n n(n-1) x^{n-1}$$

$m = n-1$

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$$k y_1 \left(\frac{4}{x} - 1\right) + k y_1' (2x - 2) + 4 \text{ series} = 0$$

$$y_1' = \sum_{n=0}^{\infty} (n+1)(n+4) x^{n+3}$$

$$\sum_{n=0}^{\infty} 4k(n+1) x^{n+3} + \sum_{n=0}^{\infty} -k(n+1) x^{n+4} + \sum_{n=0}^{\infty} 2k(n+1)(n+4) x^{n+4}$$

$n+3=m \qquad n+4=m \qquad n+4=m$

$$\sum_{n=0}^{\infty} -2k(n+1)(n+4) x^{n+3} + 4 \text{ series} = 0.$$

$n+3=m$

$$\sum_{m=3}^{\infty} 4k(m-2) x^m + \sum_{m=4}^{\infty} -k(m-3) x^m + \sum_{m=4}^{\infty} 2k(m-3) m x^m +$$

$$\sum_{m=3}^{\infty} -2k(m-2)(m+1) x^m + \sum_{m=0}^{\infty} -2 b_m x^m + \sum_{m=0}^{\infty} 3 b_{m+1} (m+1) x^m$$

$$+ \sum_{m=2}^{\infty} b_m m(m-1) x^m + \sum_{m=1}^{\infty} -(m+1) m b_{m+1} x^m = 0$$

$m=0: \quad -2b_0 + 3b_1 = 0 \rightarrow \underline{\underline{b_1 = +\frac{2}{3}b_0}}$

$m=1: \quad -2b_1 + 3b_2 2 - 2b_2 = 0 \rightarrow b_2 = \frac{2}{4}b_1 = \underline{\underline{\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}b_0 = b_2}}$

$m=2: \quad -\cancel{2b_2} + 3b_3 3 + \cancel{b_2 2} - (3)2b_3 = 0 \rightarrow \underline{\underline{b_3 = 0}}$

$m=3: \quad k\{+4 - 2(4)\} - 2\overset{=0}{b_3} + \cancel{3b_4 4} + \overset{=0}{b_3}6 - \cancel{12b_4} = 0$

$$0 = -4k \rightarrow \underline{\underline{k=0}}$$

$m \geq 4$ (with $k=0$): $\quad -2b_m + 3b_{m+1}(m+1) + b_m m(m-1) + - (m+1)m b_{m+1} = 0.$

$$b_{m+1} = \frac{(2 - m^2 + m)b_m}{3(m+1) - (m+1)m} = \frac{-[m^2 - m - 2]b_m}{(m+1)(3-m)} = \frac{+(m-2)}{+(m-3)}b_m$$

$m=4: \underline{\underline{b_5 = 2b_4}}$

$m=5: b_6 = \frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{1}b_4$

$m=6 \quad b_7 = \frac{4}{\cancel{3}}\cdot \cancel{3}\, b_4$

$\vdots$

$b_m = (m-3)b_4$

$y_2 = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \ldots$

$$\boxed{y_2 = b_0(1 + \tfrac{2}{3}x + \tfrac{1}{3}x^2) + b_4(x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + \ldots)}$$

Note $y = c_1 y_1 + y_2 =$

$c_1 x^4(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots) +$

$+ b_4 x^4(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots) +$

$b_0(1 + \frac{2}{3}x + \frac{1}{3}x^2)$

$= k_1 \overbrace{x^4(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots)}^{y_1}$

$+ k_2 \underbrace{(1 + \tfrac{2}{3}x + \tfrac{1}{3}x^2)}_{y_3}$

Note 2 linearly indep. sol'ns $y_1$ & $y_3$

$= k_1 \frac{x^4}{(x-1)^2} + k_2(1 + \frac{2}{3}x + \frac{1}{3}x^2)$ (see p. 3)

(1c) $x^2 y'' + 7xy' + 13y = 0$ ← Euler eq. $x > 0$

$y = x^m \rightarrow \cancel{x^2}\, m(m-1)\cancel{x^{m-2}} + 7\cancel{x}\, m\cancel{x^{m-1}} + 13\cancel{x^m} = 0$

$m^2 - m + 7m + 13 = 0 \rightarrow m^2 + 6m + 13 = 0$

$$m = \frac{-\cancel{6}^{\,3} \pm \sqrt{\cancel{36}^{\,9} - \cancel{4}(13)}}{\cancel{2}} = -3 \pm 2i$$

$y(x) = x^{-3}\left(c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right)$

(1d) $x^2 y'' + xy' + x^2 y = 0$ ← Bessel's eq of order 0

$y = c_1 J_0(x) + c_2 Y_0(x)$ $x > 0$

(a). $x^2 y'' + (\alpha^2\beta^2 x^{2\beta} + \frac{1}{4} - \nu^2\beta^2)\, y = 0, \; x > 0$

$y = x^{1/2} f(\alpha x^\beta)$ Let $\xi = \alpha x^\beta$

$$\frac{dy}{dx} = \left( \frac{+1}{2\sqrt{x}} f(\xi) + \cancel{x}^{1/2} \frac{df}{d\xi} \alpha\beta x^{\beta - \cancel{1}^{1/2}} \right)$$

$$\frac{d^2y}{dx^2} = \left( \frac{-1}{4x^{3/2}} f(\xi) + \frac{1}{2\cancel{\sqrt{x}}} \frac{df}{d\xi} \alpha\beta x^{\beta - \cancel{1}\frac{3}{2}} + \frac{d^2f}{d\xi^2} \alpha^2\beta^2 x^{2\beta - \frac{3}{2}} + \frac{df}{d\xi} \alpha\beta(\beta - \tfrac{1}{2}) x^{\beta - 3/2} \right)$$

sub $y''(x)$ & $y(x)$ into eq

$$\cancel{-\frac{1}{4}\sqrt{x} f(\xi)} + \frac{df}{d\xi} \alpha\beta x^{\beta + \frac{1}{2}} \left( \cancel{\tfrac{1}{2}} + \beta - \cancel{\tfrac{1}{2}} \right) + \frac{d^2f}{d\xi^2} \alpha^2\beta^2 x^{2\beta + \frac{1}{2}}$$

$$+ f(\xi)\left( \alpha^2\beta^2 x^{2\beta + \frac{1}{2}} + \cancel{\tfrac{1}{4}\sqrt{x}} - \nu^2\beta^2\sqrt{x} \right)$$

$$= \beta^2 \sqrt{x} \left\{ \xi^2 \frac{d^2f}{d\xi^2} + \xi \frac{df}{d\xi} + (\xi^2 - \nu^2) f \right\}$$ Bessel's eq

$= \beta^2\sqrt{x}\,(0)$ because $f(\xi)$ is a sol'n to Bessel's eq

$= 0$ ✓

(b). $\alpha^2\beta^2 = 4 \quad 2\beta = 2. \quad \frac{1}{4} - \nu^2\beta^2 = -2$

$\alpha = \pm 2 \quad \beta = 1 \quad \frac{1}{4} - \nu^2 = -2$

$\nu^2 = \frac{1}{4} + \frac{8}{4} = \frac{9}{4}$ not an even integer

$\nu_{\pm} = \pm\frac{3}{2} \quad 2\nu_+ = 3$

$$\underline{y = c_1 \sqrt{x}\, J_{\frac{3}{2}}(2x) + c_2\sqrt{x}\, J_{-\frac{3}{2}}(2x)}$$