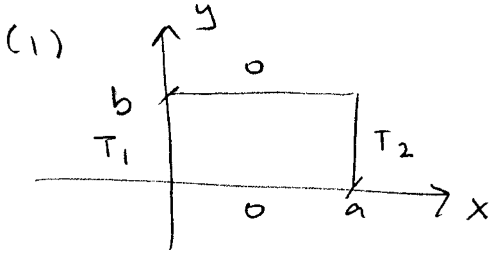
HW 14 - A Sol$^{\underline{n}}$

AEM II
Gross
S2008

(1)



$$u_{xx} + u_{yy} = 0, \quad 0<x<a, \; 0<y<b$$

$$\left.\begin{aligned} u(0,y) &= T_1 \\ u(a,y) &= T_2 \end{aligned}\right\} \; 0<y<b$$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(x,b) &= 0 \end{aligned}\right\} \; 0<x<a$$

$$u(x,y) = X(x)Y(y) \rightarrow \frac{X''Y + XY'' = 0}{XY} \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = K$$

(b) $X'' - KX = 0$, (a) $Y'' + KY = 0$, $Y(0)=0,\ Y(b)=0$ $\rightarrow K = \left(\frac{n\pi}{b}\right)^2$

$Y = \sin\frac{n\pi y}{b}$, $n = 1,2,3,\dots$

$X = e^{rx} \rightarrow$

$r^2 = K = \left(\frac{n\pi}{b}\right)^2 \rightarrow r = \pm n\pi/b$

$X = c_1 \cosh\frac{n\pi x}{b} + c_2 \sinh\frac{n\pi x}{b}$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b}\cosh\frac{n\pi x}{b} + B_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi x}{b}}$$

$$u(0,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b} = T_1 \Rightarrow$$

$$A_n = \frac{\langle T_1, \sin\frac{n\pi y}{b}\rangle}{\langle \sin\frac{n\pi y}{b}, \sin\frac{n\pi y}{b}\rangle} \qquad (*)$$

$$u(a,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b}\cosh\frac{n\pi a}{b} + B_n \sin\frac{n\pi y}{b}\sinh\frac{n\pi a}{b} = T_2$$

$$\Rightarrow \left[A_n \cosh\frac{n\pi a}{b} + B_n \sinh\frac{n\pi a}{b}\right] = \frac{\langle T_2, \sin\frac{n\pi y}{b}\rangle}{\langle \sin\frac{n\pi y}{b}, \sin\frac{n\pi y}{b}\rangle}$$

$(*) \rightarrow$

$$A_n = \frac{2T_1}{b}\int_0^b \sin\frac{n\pi y}{b}\,dy = -\frac{2T_1}{\cancel{b}}\frac{\cancel{b}}{n\pi}\cos\frac{n\pi y}{b}\Bigg|_{y=0}^{b}$$

$$= -\frac{2T_1}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ even} \\ \frac{4T_1}{n\pi}, & n \text{ odd} \end{cases}$$

$$A_n \cosh\frac{n\pi a}{b} + B_n \sinh\frac{n\pi a}{b} = \frac{2}{b} T_2 \int_0^b \sin\frac{n\pi y}{b}\, dy$$

$$\downarrow$$

$$-\frac{2T_1}{n\pi}\left[(-1)^n - 1\right] \qquad\qquad = -\frac{2T_2}{n\pi}\left[(-1)^n - 1\right]$$

$$\rightarrow \quad B_n = \frac{-\frac{2}{n\pi}\left[(-1)^n - 1\right]\left[T_2 - T_1 \cosh\frac{n\pi a}{b}\right]}{\sinh\frac{n\pi a}{b}} = \begin{cases} 0, & n \text{ even} \\ \end{cases}$$

$$\text{So } u(x,y) = \sum_{n=0}^{\infty} A_{2n+1} \sin\frac{(2n+1)\pi y}{b} \cosh\frac{(2n+1)\pi x}{b}$$

$$+ B_{2n+1} \sin\frac{(2n+1)\pi y}{b} \sinh\frac{(2n+1)\pi x}{b}$$

$$= \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi y}{b}\left[A_{2n+1}\cosh\frac{(2n+1)\pi x}{b} + B_{2n+1}\sinh\frac{(2n+1)\pi x}{b}\right]$$

$$= \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi y}{b}\left[\frac{-4T_1}{(2n+1)\pi}\cosh\frac{(2n+1)\pi x}{b} + \frac{-\frac{4}{(2n+1)\pi}\left[T_2 - T_1\cosh\frac{n\pi a}{b}\right]}{\sinh\frac{(2n+1)\pi a}{b}} \cdot \sinh\frac{(2n+1)\pi x}{b}\right]$$

(2) $u_{tt} = c^2(u_{xx} + u_{yy})$

$u(0,y,t) = 0$

$u(a,y,t) = 0$

$u(x,0,t) = 0$

$u(x,b,t) = 0$

$u(x,y,0) = f(x,y)$

$u_t(x,y,0) = 0$

$u(x,y,t) = X(x)Y(y)T(t)$

$$\frac{XYT'' = c^2(X''YT + XY''T)}{XYT}$$

$$\frac{T''}{T} = c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = -k$$

$T'' + kT = 0$.

$$c^2\frac{X''}{X} = -c^2\frac{Y''}{Y} - k = -\sigma \rightarrow X'' + \frac{\sigma}{c^2}X = 0.$$

$$\rightarrow Y'' + \left(\frac{k-\sigma}{c^2}\right)Y = 0.$$

$X(0)=0$ $\quad$ $Y(0)=0$ $\quad$ $T'(0)=0$.

$X(a)=0$ $\quad$ $Y(b)=0$.

we've solved the S-L prob in X before

$\frac{\sigma}{c^2} > 0 \rightarrow \frac{\sigma}{c^2} = \lambda^2$, $\lambda^2 = \left(\frac{n\pi}{a}\right)^2$, $X(x) = \sin\left(\frac{n\pi x}{a}\right)$

S-L prob. in Y goes the same way:

$\frac{k-\sigma}{c^2} = \frac{k - \lambda^2c^2}{c^2} > 0 \rightarrow k > \lambda^2 \rightarrow k > 0 \rightarrow k = \mu^2$

$\frac{\mu^2 - \lambda^2c^2}{c^2} = \left(\frac{m\pi}{b}\right)^2$, $Y(y) = \sin\left(\frac{m\pi x}{b}\right)$, $m = 1, 2, \ldots$

$T'' + kT = 0$, $\quad T = e^{rt} \rightarrow r^2 = -k = -\mu^2 \rightarrow r = \pm\mu i$

$T(t) = c_1\sin(\mu t) + c_2\cos(\mu t)$.

$T'(0) = c_1\mu\cos(0) - c_2\mu\sin(0) = 0$ if $c_1 = 0$.

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cos(\mu t),$$

$$\mu = \sqrt{\left(\frac{cm\pi}{b}\right)^2 + \lambda^2c^2}$$

$$\rightarrow \mu = \sqrt{\left(\frac{cm\pi}{b}\right)^2 + \left(\frac{n\pi c}{a}\right)^2}$$

$$\rightarrow \mu = c\pi\sqrt{\left(\frac{m}{b}\right)^2 + \left(\frac{n}{a}\right)^2}$$

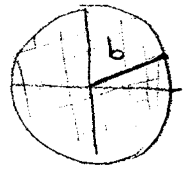
$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{b}\right) = f(x,y)$$

$$C_{nm} = \frac{\int_0^b\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{b}\right)dx\,dy}{\int_0^b\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\sin^2\left(\frac{m\pi x}{b}\right)dx\,dy}$$

$\left(\text{denom} = \frac{ab}{4}\right).$

Ex Find steady-state temp. dist$^{\underline{n}}$ in a disk of radius b w/ edge maintained at $f(\theta)$ degrees ($f(\theta)$ 2π-per$^{\underline{ic}}$).

(1) $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$



$0 < r < b$

seek $u(r,\theta)$. $\quad 0 < \theta < 2\pi$

(2) $u(b,\theta) = f(\theta)$

(3) $u(0,\theta)$ bounded

periodic conditions in θ:

(4) $u(r,0) = u(r,2\pi)$

(5) $u_\theta(r,0) = u_\theta(r,2\pi)$

$u(r,\theta) = R(r)\Theta(\theta)$

$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$

$R(b)\Theta(\theta) = f(\theta) \rightsquigarrow$ no concl.

$R(0)\Theta(\theta)$ bounded $\rightsquigarrow$ $R(0)$ bounded

$\cancel{R(r)}\Theta(0) = \cancel{R(r)}\Theta(2\pi)$

$\Theta'(0) = \Theta'(2\pi)$

As in class,

$$-r^2\frac{R''}{R} - r\frac{R'}{R} = +\frac{\Theta''}{\Theta} = -k$$

$r^2R'' + rR' - kR = 0$, $\quad \Theta'' + k\Theta = 0$

$R(0)$ bounded $\quad \left.\begin{array}{l}\Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi)\end{array}\right]$ periodic S-L prob

$\Theta = e^{r\theta} \to r^2 = -k \to r = \pm\sqrt{-k}$

(I) $k=0$: $\Theta(\theta) = c_1 + c_2\theta$

$\Theta(0) = c_1 = \Theta(2\pi) = c_1 + c_2 2\pi \Rightarrow c_2 = 0$

$\Theta'(0) = \Theta'(2\pi) = 0$ ✓

$\boxed{\Theta_0(\theta) \equiv 1 \text{ is an e-fnc corr'ing to } k_0 = 0.}$

(II) $k<0$: $\Theta(\theta) = c_1 e^{\sqrt{-k}\,\theta} + c_2 e^{-\sqrt{-k}\,\theta}$

$\Theta(0) = c_1 + c_2 = \Theta(2\pi) = c_1 e^{\sqrt{-k}\,2\pi} + c_2 e^{-\sqrt{-k}\,2\pi}$

$\Theta'(0) = c_1\sqrt{-k} + c_2\sqrt{-k} = \Theta'(2\pi) =$

$= c\sqrt{-k}\, e^{\sqrt{-k}\,2\pi} - c_2 e^{-\sqrt{-k}\,2\pi}\sqrt{-k}$

$$\begin{aligned}
&[c_1 + c_2 = c_1 e^{\sqrt{-k}\,2\pi} + c_2 e^{-\sqrt{-k}\,2\pi}]\sqrt{-k} \\
+\ &c_1\sqrt{-k} - c_2\sqrt{-k} = c_1\sqrt{-k}\, e^{\sqrt{-k}\,2\pi} - c_2\sqrt{-k}\, e^{-\sqrt{-k}\,2\pi} \\
\hline
&\cancel{2}\cancel{c_1}\sqrt{-k} = \cancel{2}\cancel{c_1}\sqrt{-k}\, \underbrace{e^{\sqrt{-k}\,2\pi}}_{>0} \rightarrow \text{no sol if } c_1 \neq 0
\end{aligned}$$

similarly, get $c_2 = 0$ $\therefore$ evals k not neg.

(III) $k = \lambda^2$: $\Theta(\theta) = c_1 \sin(\lambda\theta) + c_2\cos(\lambda\theta)$

① $\Theta(0) = c_2 = \Theta(2\pi) = c_1\sin(2\pi\lambda) + c_2\cos(2\pi\lambda)$.

② $\Theta'(0) = c_1\lambda\cos(0) - c_2\lambda\sin(0) = c_1\lambda = \Theta'(2\pi) =$

$= c_1\lambda\cos(2\pi\lambda) - c_2\lambda\sin(2\pi\lambda) \Rightarrow$

$$\begin{bmatrix} \sin 2\pi\lambda & \cos 2\pi\lambda - 1 \\ \cos 2\pi\lambda - 1 & -\sin 2\pi\lambda \end{bmatrix} \underline{c} = \underline{0}$$

$\not\Rightarrow \underline{c} = \underline{0}$ if det $=0$, i.e. $-\sin^2 2\pi\lambda - \cos^2 2\pi\lambda - 1 + 2\cos 2\pi\lambda = 0$

$\leftrightarrow 2 - 2\cos 2\pi\lambda = 0 \leftrightarrow$

$\cos 2\pi\lambda = 1$ if $\boxed{\lambda = 1, 2, 3, \ldots}$ sat ① & ②.

$\Theta(\theta) = c_1\sin(n\theta) + c_2\cos(n\theta)$, $n = 1, 2, 3, \ldots$

solved $r^2R'' + rR' - \lambda^2 R = 0$ in previous problem →.

$R(r) = c_1 r^{\lambda} + c_2 r^{-\lambda}$

$R(0)$ bounded $\rightarrow c_2 = 0$. $(\lambda > 0)$

$R(r) = r^{+\lambda} \rightarrow R(r) = r^{+n}$, $n = 1, 2, 3, \ldots$

$u_n(r,\theta) = r^{+n}\{c_1\sin(n\theta) + c_2\cos n\theta)\}$, $n = 1, 2, \ldots$

$u_0(r,\theta) = r^0\Theta_0(\theta) \doteq 1$ all satisfy (1)-(2), (4)-(5),

a linear "homogeneous" BVP $\therefore$ superposition is also a solution ...

$$u(r,\theta) = c_0 + \sum_{n=1}^{\infty} r^{+n}\{a_n \sin n\theta + b_n \cos n\theta\}$$

Impose condition (2) →

$$u(b,\theta) = c_0 + \sum_{n=1}^{\infty} b^{+n}\{a_n \sin n\theta + b_n \cos n\theta\} = f(\theta)$$

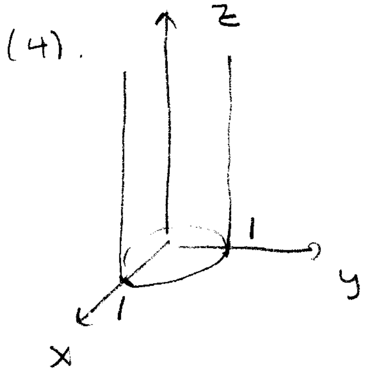
$$c_0 = \frac{\int_0^{2\pi} f(\theta)\cdot 1 \, d\theta}{\int_0^{2\pi} 1 \cdot d\theta} \rightarrow c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$b^{+n} a_n = \frac{\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta}{\int_0^{2\pi} \sin^2 n\theta \, d\theta} \rightarrow a_n = \frac{1}{b^n \pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

$$b^{+n} b_n = \frac{\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta}{\int_0^{2\pi} \cos^2 n\theta \, d\theta} \rightarrow b_n = \frac{1}{b^n \pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

In our problem, $f(\theta) = \cos\theta$, $b = 5$.

(4).



$u_t = k\left[u_{rr} + \frac{1}{r} u_r\right]$, $0 < r < 1$, $t > 0$ seek $u(r,t)$

$\lim_{r \to 0^+} |u(r,t)| < \infty$, $u(1,t) = 0$

$u(r,0) = f(r)$

$$u(r,t) = R(r)T(t) \rightarrow \frac{RT' = k\left(R''T + \frac{1}{r} R'T\right)}{kRT} \rightarrow$$

$$\frac{T'}{kT} = \frac{R'' + \frac{1}{r} R'}{R} = -\delta \rightarrow$$

$R'' + \frac{1}{r} R' + \delta R = 0$, $\lim_{r \to 0^+} |R| < \infty$, $R(1) = 0$

$\downarrow$

Solved in class to get

$\delta = \beta_n^2 \leftarrow [\text{roots of } J_0(x)]^2$

$R(r) = J_0(\beta_n r)$, $n = 1, 2, 3, \dots$

$T' + \delta k T = 0$

$T = e^{rt}$

$r = -\delta k$

$T = e^{-\beta_n^2 k t}$

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\beta_n r) e^{-\beta_n^2 k t}}$$

$$u(r,0) = \sum_{n=1}^{\infty} a_n J_0(\beta_n r) = f(r)$$

$$\boxed{a_n = \frac{\int_0^1 f(r) J_0(\beta_n r) r\,dr}{\int_0^1 [J_0(\beta_n r)]^2 r\,dr}}$$