Hw 13 — A Sol$^{\underline{n}}$

AEM IV
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S2008.

(1). without forcing term $F(x,t)$, IBVP has sol$^{\underline{n}}$ (see notes):

$$y_h = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}, \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

with the nonhomog. PDE (where source has t-dependence) seek $\boxed{y = \sum_{n=1}^{\infty} T_n(t) \sin\frac{n\pi x}{L}}$. Find $T_n(t)$.

This y satisfies the BC$^{\underline{s}}$ $y(0,t) = y(L,t) = 0$.

Note $y_t = \sum_{n=1}^{\infty} T_n'(t) \sin\frac{n\pi x}{L}$

To satisfy the IC$^{\underline{s}}$ need

$y(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin\frac{n\pi x}{L} \doteq f(x)$. we get this if

$$\underline{T_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}$$

& $y_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) \sin\frac{n\pi x}{L} = 0$. we get this if

$$T_n'(0) = \frac{2}{L}\int_0^L 0 \cdot \sin\frac{n\pi x}{L} dx$$

i.e. if $\underline{T_n'(0) = 0}$.

Sub y into $y_{tt} - c^2 y_{xx} = F(x,t)$:

$$y_{tt} = \sum_{n=1}^{\infty} T_n''(t) \sin\frac{n\pi x}{L}$$

$$-c^2 y_{xx} = \sum_{n=1}^{\infty} c^2 \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin\frac{n\pi x}{L}.$$

$$F(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L} \longleftarrow$$

eigenfnc expan of F:

$$\boxed{B_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi x}{L}dx}$$

equating like terms $\Rightarrow$.

$$T_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 T_n(t) = B_n(t).$$

Recall IC's found above: $T_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$,

$T_n'(0) = 0.$

Solve the IVP by variation of parameters:

$$T_{n\,homog} = T_{nh} = e^{rt} \Rightarrow r^2 + \left(\frac{cn\pi}{L}\right)^2 = 0 \Rightarrow$$

$$T_{nh} = c_1 \sin\frac{n\pi ct}{L} + c_2\cos\frac{n\pi ct}{L}$$

$$T_{np} = u(t)\sin\frac{n\pi ct}{L} + v(t)\cos\frac{n\pi ct}{L},$$

where $u(t) = \int_0^t -\frac{B_n(\tau)}{W}\cos\frac{cn\pi\tau}{L}\,d\tau$,

$v(t) = \int_0^t \frac{B_n(\tau)}{W}\sin\frac{cn\pi\tau}{L}\,d\tau$, &

$$W = \begin{vmatrix} \sin\frac{cn\pi t}{L} & \cos\frac{n\pi ct}{L} \\ +\frac{cn\pi}{L}\cos\frac{cn\pi t}{L} & -\frac{cn\pi}{L}\sin\frac{n\pi ct}{L} \end{vmatrix}$$

$$= -\frac{cn\pi}{L}\underbrace{[\sin^2 + \cos^2]}^{1}$$

$$\boxed{\begin{aligned} T_n(t) = {} & \underset{0\ (\text{see below})}{\cancel{c_1}}\sin\frac{cn\pi t}{L} + \underset{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\ (\text{see below})}{\cancel{c_2}}\cos\frac{cn\pi t}{L} + \\ & + \sin\frac{cn\pi t}{L}\left\{\int_0^t B_n(\tau)\cos\frac{cn\pi\tau}{L}\,\frac{L}{cn\pi}\,d\tau\right\} + \\ & + \cos\frac{cn\pi t}{L}\left\{\int_0^t -B_n(\tau)\sin\frac{cn\pi\tau}{L}\,\frac{L}{cn\pi}\,d\tau\right\} \end{aligned}}$$

Apply IC's from top of page:

$T_n(0) = \underline{c_2 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx}$

$T_n'(0) = c_1\frac{cn\pi}{L}\cos(0) = 0 \to \underline{c_1 = 0}.$

Now boxed sol$^{\underline{n}}$ y on p. 1 is fully determined.

(2) Heat conduction on an infinite rod :

$u_t = Ku_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$, $-\infty < x < \infty$

Assume $\lim_{x\to\pm\infty} u(x,t) = 0$, $\lim_{x\to\pm\infty} u_x(x,t) = 0$

mult. by $e^{-i\omega x}$ & integrate with respect to x.

$$\int_{-\infty}^{\infty} u_t e^{-i\omega x}\,dx = K\int_{-\infty}^{\infty} \underbrace{u_{xx}}_{dv} \overbrace{e^{-i\omega x}}^{u}\,dx$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u e^{-i\omega x}\,dx = K\left[e^{-i\omega x} u_x \Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} \underbrace{u_x}_{dv} \overbrace{e^{-i\omega x}}^{u}\,dx\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\hat{u}(\omega,t) = K\left[i\omega e^{-i\omega x} u\Big|_{-\infty}^{\infty} + (i\omega)^2\int_{-\infty}^{\infty} u e^{-i\omega x}\,dx\right]$$

(1) $\boxed{\frac{\partial}{\partial t}\hat{u}(\omega,t) = -K\omega^2\hat{u}(\omega,t)}$ $\leftarrow$ «ODE»

Transform the I.C. :

$$\int_{-\infty}^{\infty} u(x,0)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$\Rightarrow$ (2) $\boxed{\hat{u}(\omega,0) = \hat{f}(\omega)}$ (1) & (2) is the transformed problem.

$\hat{u} = ce^{-K\omega^2 t}$

$\hat{u}\Big|_{t=0} = c = \hat{f}(\omega) \Rightarrow \underbrace{\hat{u}(\omega,t) = \hat{f}(\omega)e^{-K\omega^2 t}}_{\text{sol'n to transformed prob.}}$

$u(x,t) = \mathcal{F}^{-1}\{\hat{f}(\omega)e^{-K\omega^2 t}\}$.

Recall $\mathcal{F}^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\} = f(x) * g(x)$. convolution thm.

Note $\mathcal{F}^{-1}\{e^{-K\omega^2 t}\} = \frac{1}{2\sqrt{\pi K t}} e^{-x^2/4Kt}$

$$u(x,t) = f(x) * \frac{1}{2\sqrt{\pi K t}} e^{-x^2/4Kt}$$

$$= \frac{1}{2\sqrt{\pi K t}}\int_{-\infty}^{\infty} f(\xi)e^{-\frac{(x-\xi)^2}{4Kt}}\,d\xi \quad , t > 0$$

$$= \frac{1}{2\sqrt{\pi K t}}\int_{-1}^{1} e^{-\xi} e^{-(x-\xi)^2/4Kt}\,d\xi, \quad t > 0$$

(2) cont'd.

> $u := t \rightarrow \dfrac{1}{2\cdot\text{sqrt}(\text{Pi}\cdot t)}\cdot int\left(\exp\left(-\text{xi}-\dfrac{(x-\text{xi})^2}{4\cdot t}\right), \text{xi}=-1..1\right);$

$$u := t \rightarrow \frac{1}{2}\,\frac{\int_{-1}^{1} e^{-\xi-\frac{1}{4}\frac{(x-\xi)^2}{t}}\,d\xi}{\sqrt{\pi t}} \tag{1}$$

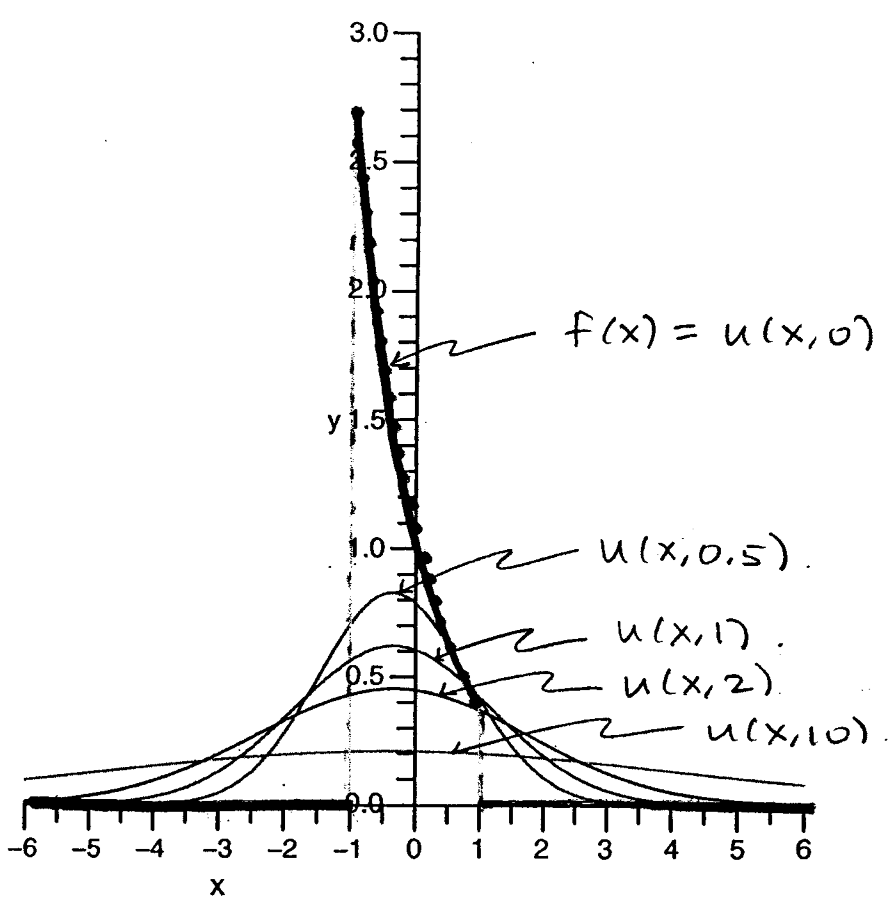
> $int\left(\exp\left(-\xi-\dfrac{(x-\xi)^2}{4\cdot t}\right), \xi=-1..1\right);$

$$-\sqrt{\pi}\,e^{t-x}\sqrt{t}\left(-\text{erf}\left(\frac{1}{2}\,\frac{1-2\,t+x}{\sqrt{t}}\right)+\text{erf}\left(\frac{1}{2}\,\frac{-1-2\,t+x}{\sqrt{t}}\right)\right) \tag{2}$$

> $f(x) := piecewise(x > -1 \text{ and } x < 1, \exp(-x));$

$$f := x \rightarrow piecewise(-1 < x \textbf{ and } x < 1, e^{-x}) \tag{3}$$

> $plot(\{f(x), u(0.5), u(1), u(2), u(10)\}, x=-6..6, y=0..3);$



As time goes on, temperature distribution spreads & flattens.

> ?piecewise

>

3

Heat conduction on a semi-infinite rod.

$u_t = \kappa u_{xx}$, $x > 0$, $t > 0$.

$u(0,t) = h(t)$, $t > 0$

$\lim_{x\to\infty} u(x,t) = 0$, $t > 0$ } BCs.

$u(x,0) \equiv 0$, $x > 0$. } IC.

multiply by e^{-st}, & integrate with respect to t.

$\int_0^\infty \underbrace{u_t e^{-st}\, dt}_{dv} = \kappa \int_0^\infty u_{xx} e^{-st}\, dt$

$e^{-st} u \Big|_0^\infty + s \int_0^\infty u e^{-st}\, dt = \kappa \frac{\partial^2}{\partial x^2} \int_0^\infty u e^{-st}\, dt$

$\underbrace{-u(x,0)}_{=0} + s\, U(x,s) = \kappa \frac{\partial^2}{\partial x^2} U(x,s)$, $U(x,s) = \mathcal{L}\{u(x,t)\}$

$s U = \kappa U_{xx}$. (1)

Transform the BCs:

$\int_0^\infty u(0,t)\, e^{-st}\, dt = \int_0^\infty h(t) e^{-st}\, dt$

$\rightarrow$ $U(0,s) = H(s)$ (2)

$\int_0^\infty \lim_{x\to\infty} u(x,t)\, e^{-st}\, dt = 0 \rightarrow \lim_{x\to\infty} U(x,s) = 0$ (3)

(1)-(3) is the transformed problem

$U = e^{rx} \rightarrow s = \kappa r^2 \rightarrow r = \pm\sqrt{s/\kappa}$ $\left\{ \begin{array}{l} s > 0, \\ \kappa > 0. \end{array}\right.$

$\rightarrow$ $U = c_1 e^{\sqrt{s/\kappa}\, x} + c_2 e^{-\sqrt{s/\kappa}\, x}$

BC (3) $\rightarrow$ $c_1 = 0$

$U(0,s) = c_2 = H(s) \rightarrow \underbrace{U(x,s) = H(s) e^{-\sqrt{s/\kappa}\, x}}$

sol'n to the transformed problem

$u(x,t) = \mathcal{L}^{-1}\{ H(s) e^{-\sqrt{s/\kappa}\, x} \}$

Recall $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$ convolution thm

Note $\mathcal{L}^{-1}\{e^{-\sqrt{s}\cdot a}\} = \frac{a}{2\sqrt{\pi}\, t^{3/2}} e^{-a^2/4t}$, $a = \frac{x}{\sqrt{K}}$

$$u(x,t) = h(t) * \frac{x}{2\sqrt{\pi K}\, t^{3/2}} e^{-x^2/4Kt}$$

$$= \int_0^t h(\tau) \frac{x}{2\sqrt{\pi K}\,(t-\tau)^{3/2}} \exp\left[\frac{-x^2}{4K(t-\tau)}\right] d\tau$$

$$= \frac{x}{2\sqrt{\pi K}} \int_0^t h(\tau) \frac{1}{(t-\tau)^{3/2}} \exp\left[\frac{-x^2}{4K(t-\tau)}\right] d\tau,$$

$x > 0$.

8. $u(x,t) = h(t)$ when $x = 0$.

$$u(x,t) = \frac{x}{2\sqrt{\pi K}} \int_0^t \tau^2 \frac{1}{(t-\tau)^{3/2}} \exp\left(\frac{-x^2}{4K(t-\tau)}\right) d\tau.$$

(4) $y_{tt} = c^2 y_{xx}, \quad x>0,\ t>0$ HW13, p7

$y(0,t) = \begin{cases} \sin(2\pi t), & 0 \le t \le 1 \\ 0, & t>1 \end{cases}$ $\quad \lim_{x\to\infty} y(x,t)=0, \quad t>0$

$y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0, \quad x>0 \quad \Big\} (*)$

$$\int_0^\infty \frac{\partial^2 y}{\partial t^2} e^{-st}\, dt = c^2 \int_0^\infty \frac{\partial^2 y}{\partial x^2} e^{-st}\, dt$$

$\downarrow$ int by parts twice & use $(*)$

$$s^2 Y(x,s) = c^2 \frac{\partial^2}{\partial x^2} Y(x,s).$$

$$\int_0^\infty y(0,t) e^{-st}\, dt = \int_0^1 \sin 2\pi t\ e^{-st}\, dt$$

$Y(0,s) = \frac{2\pi}{s^2+4\pi^2}(1-e^{-s})$ BC(1), $\quad \lim_{x\to\infty} Y(x,s) = 0$ BC(2).

$Y = e^{rx} \to s^2 = c^2 r^2 \to r = \pm \frac{s}{c}$

$Y(x,s) = c_1 e^{sx/c} + c_2 e^{-sx/c}$, $\quad s>0,\ c>0$.

BC (2) $\to$ $c_1 = 0$

BC (1) $\to$ $c_2 = \frac{2\pi}{s^2+4\pi^2}(1-e^{-s})$.

$$Y(x,s) = \frac{2\pi}{s^2+4\pi^2}(1-e^{-s})\, e^{-sx/c}$$

$y(x,t) = \mathcal{L}^{-1}\{Y(s)\}$ as above

$$= \mathcal{L}^{-1}\left\{\frac{2\pi}{s^2+4\pi^2} e^{-sx/c}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{2\pi}{s^2+4\pi^2} e^{-s(1+x/c)}\right\}$$

$(\star)$

$= H(t-\frac{x}{c}) \sin(2\pi(t-\frac{x}{c})) - H(t-1-\frac{x}{c}) \sin(2\pi(t-1-\frac{x}{c}))$

optional (maple)

Addendum:

Note we could denote the B.C. involving the sine fnc as

$$y(0,t) = f(t).$$

The transformed BC can be denoted

$$Y(0,s) = F(s).$$

Then $Y(x,s) = c_2 e^{-sx/c} \Rightarrow c_2 = F(s)$.

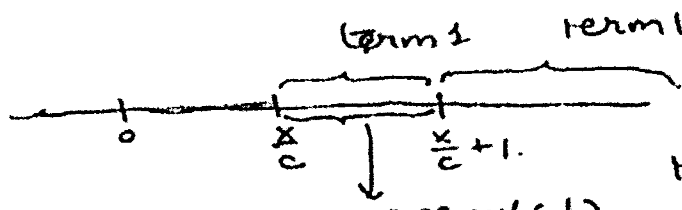
$$\Rightarrow y(x,t) = \mathcal{L}^{-1}\{F(s)\,e^{-sx/c}\}$$
$$= H(t-\tfrac{x}{c})\, f(t-\tfrac{x}{c}) \quad \text{by a Laplace Transform Thm}$$

$$= \begin{cases} H(t-\frac{x}{c}) \sin(2\pi(t-\frac{x}{c})) & \text{if } 0 \le t-\frac{x}{c} \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$0 \le t \le 1+\frac{x}{c}$
& $t \ge \frac{x}{c}$.

$$(\dagger) = \begin{cases} \sin(2\pi(t-\frac{x}{c})) & \text{if } \frac{x}{c} \le t \le 1+\frac{x}{c} \\ 0 & \text{otherwise} \end{cases}$$

which is equivalent to $(*) = \text{term1} - \text{term2}$:



agrees w/ (†).

term2 : $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha \rightarrow$
term2 $= \sin(2\pi(t-\frac{x}{c}))\cdot 1\ \cancel{\cos(2\pi)} +$
$\cancel{\sin(2\pi(t-\frac{x}{c}))\cdot\sin(-2\pi)}\ 0 \Rightarrow$
term1 − term2 = 0 for $t > \frac{x}{c}+1$

15: $\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t)\, dx =$

$144 \{ e^{-i\omega x} \frac{\partial u}{\partial x} \Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-i\omega x} dx \}$

$\Rightarrow \frac{\partial^2}{\partial t^2} \hat{u}(\omega,t) = 144 i\omega \left[e^{-i\omega x} u \Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} u e^{-i\omega x} dx \right]$

if we assume $\lim_{x \to \pm\infty} \frac{\partial u}{\partial x} = 0$.

$\Rightarrow \frac{\partial^2 \hat{u}}{\partial t^2}(\omega,t) = -144\,\omega^2 \cdot \hat{u}(\omega,t)$. (a).

if we assume $\lim_{x \to \pm\infty} u = 0$.

Take F.T. of IC's:

IC1: $\int_{-\infty}^{\infty} e^{-i\omega x} u(x,0)\, dx = \int_{-\infty}^{0} e^{+5x - i\omega x} dx + \int_{0}^{\infty} e^{-5x - i\omega x} dx$.

$\Rightarrow \hat{u}(\omega,0) = \frac{1}{5 - i\omega} e^{(5-i\omega)x} \Big|_{-\infty}^{0} + \frac{1}{-5-i\omega} e^{-(5+i\omega)x} \Big|_{0}^{\infty}$

$= \frac{1}{5-i\omega} \lim_{b \to \infty} \left[e^{0} - \cancel{e^{5b}}^{\,0} (\cos \omega x - i \sin \omega x) \right] + \frac{1}{-5-i\omega} \lim_{b \to \infty} \left[\cancel{e^{-5b}}^{\,0} (\cos \omega x - i \sin \omega x) - e^{0} \right]$

$= \frac{1}{5-i\omega} + \frac{1}{5+i\omega} = \frac{5 + \cancel{i\omega} + 5 - \cancel{i\omega}}{25 + \omega^2}$

$\Rightarrow \hat{u}(\omega,0) = \frac{10}{25+\omega^2}$; IC2: $\frac{\partial}{\partial t} \hat{u}(\omega,0) = 0$. (b,c)

(5) cont.

Solve the BVP (a)-(b,c) :

Let $\hat{u} = e^{rt}$ $\Rightarrow$ $r^2 = -144\omega^2$ $\rightarrow$ $r = \pm 12\omega i$

$\rightarrow$ $\hat{u} = c_1 \cos 12\omega t + c_2 \sin 12\omega t$

$\hat{u}(\omega, 0) = c_1 = \dfrac{10}{25+\omega^2}$

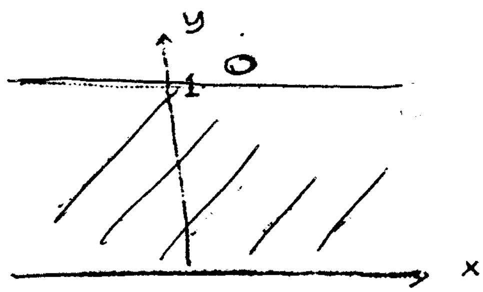
$\dfrac{\partial \hat{u}}{\partial t} = -12 c_1 \omega \sin 12\omega t + 12 c_2 \omega \cos 12\omega t$

$\dfrac{\partial \hat{u}}{\partial t}(\omega, 0) = 12 c_2 \omega = 0 \Rightarrow c_2 = 0.$

$\rightarrow$ $\hat{u}(\omega, t) = \dfrac{10}{25+\omega^2} \cos 12\omega t$

$\rightarrow$ $u(x,t) = \mathcal{F}^{-1}\left\{ \dfrac{10}{5+\omega^2} \cos 12\omega t \right\}$

$= \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{10}{5+\omega^2} \cos 12\omega t \, e^{i\omega x} \, d\omega.$ //



(6) $u_{xx}+u_{yy}=0$.

$$u(x,0)=\begin{cases}0, & x<0\\ e^{-ax}, & x>0\end{cases}$$

$u(x,1)=0$ Take F.T. $\Rightarrow$.

$$\int_{-\infty}^{\infty} u_{xx}\, e^{-i\omega x}\,dx = -\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial y^2}\, e^{-i\omega x}\,dx$$

$$\underbrace{e^{-i\omega x}u_x\Big|_{-\infty}^{\infty}}_{0} + i\omega\int_{-\infty}^{\infty} u_x\, e^{-i\omega x}\,dx = -\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} u\, e^{-i\omega x}\,dx$$

Assuming $u_x\to 0$ as $x\to\pm\infty$

$$i\omega\Big\{\underbrace{u\,e^{-i\omega x}\Big|_{-\infty}^{\infty}}_{0} + i\omega\int_{-\infty}^{\infty} u\, e^{-i\omega x}\,dx\Big\} = -\frac{\partial^2}{\partial y^2}\hat{u}(\omega,y)$$

Assuming $u\to 0$ as $x\to\pm\infty$

$$+\omega^2\,\hat{u}(\omega,y) = +\frac{\partial^2}{\partial y^2}\hat{u}(\omega,y) \qquad ①$$

F.T. of BC's : $\int_{-\infty}^{\infty} u(x,0)\,e^{-i\omega x}\,dx = \int_{0}^{\infty} e^{-ax}\,e^{-i\omega x}\,dx \rightarrow$

$$\hat{u}(\omega,0) = \frac{e^{-x(a+i\omega)}}{a+i\omega}\Bigg|_{x=0}^{\infty} = \frac{\lim_{x\to\infty}\Big[\underbrace{e^{-ax}}_{\text{ampl}\to 0}\,\overbrace{e^{-i\omega x}}^{\text{osc.}}\Big]-1}{a+i\omega} \rightarrow$$

$$\hat{u}(\omega,0) = \frac{-1}{a+i\omega}\,\frac{(a-i\omega)}{(a-i\omega)}$$

or $\hat{u}(\omega,0) = \dfrac{-a+i\omega}{a^2+\omega^2}$

$$\int_{-\infty}^{\infty} u(x,1)\,e^{-i\omega x}\,dx = 0 \rightarrow \hat{u}(\omega,1)=0$$

solving ① $\rightarrow \hat{u}=e^{ry} \rightarrow r^2=\omega^2 \rightarrow r=\pm\omega$.

$$\hat{u} = c_1 e^{\omega y} + c_2 e^{-\omega y}$$

(6 cont'd)

$$\hat{u}(\omega,0) = c_1 + c_2 = \frac{-a+i\omega}{a^2+\omega^2}$$

$$\hat{u}(\omega,1) = c_1 e^{\omega} + c_2 e^{-\omega} = 0 \rightarrow c_2 = -c_1 e^{\omega 2}$$

$$\rightarrow c_1(1-e^{2\omega}) = \frac{-a+0\omega}{a^2+\omega^2} \rightarrow c_1 = \frac{-a+i\omega}{(a^2+\omega^2)(1-e^{2\omega})}$$

$$c_2 = \frac{e^{2\omega}(a-i\omega)}{(a^2+\omega^2)(1-e^{2\omega})}$$

$$\hat{u}(\omega,y) = \frac{a-i\omega}{(a^2+\omega^2)(1-e^{2\omega})}\left[-e^{\omega y} + e^{2\omega}e^{-\omega y}\right]$$

$$u(x,y) = \mathcal{F}^{-1}\{\hat{u}(\omega,y)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\frac{a-i\omega}{(a^2+\omega^2)(1-e^{2\omega})}\left(-e^{\omega y} + e^{\omega(2-y)}\right)\right]d\omega$$

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