HW 12, A Sol^{n}

AEM II
Gross
Sp 2008

(1) $u(x,t) = \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$

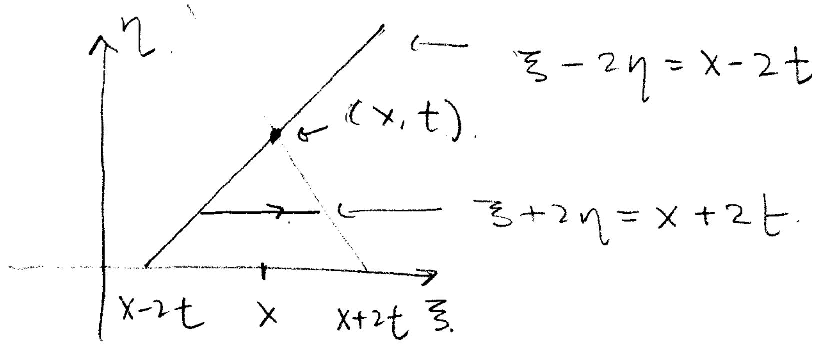
$+ \iint_{\triangle} F(\xi,\eta)\,d\xi\,d\eta$, $c=2$.

(i) $\frac{1}{2}\left[\sin(x-2t)+\sin(x+2t)\right]$ note $=$

$\frac{1}{2}\left[\sin x\cos 2t - \cos x \sin 2t\right.$
$\left. + \sin x \cos 2t + \cos x \sin 2t\right] = \sin x \cos 2t.$

(ii) $\frac{1}{2c}\int_{x-2t}^{x+2t} 2\tau\,d\tau = \frac{1}{4}\tau^2\Big|_{x-2t}^{x+2t} = \frac{1}{4}\{x^2+4xt+4t^2 - [x^2-4xt+4t^2]\} = 2xt$

(iii)



$$\int_0^t\int_{x-2t+2\eta}^{x+2t-2\eta} 2\xi\eta\,d\xi\,d\eta = \int_0^t \eta\xi^2\Big|_{\xi=x-2t+2\eta}^{x+2t-2\eta} d\eta =$$

$$= \int_0^t \eta\left[(x+2t-2\eta)^2 - (x-2t+2\eta)^2\right] d\eta$$

$$= \int_0^t \eta\left[(x+2(t-\eta))^2 - (x-2(t-\eta)^2\right] d\eta$$

$$= \int_0^t \eta\left[x\,8(t-\eta)\right] d\eta = 8\int_0^t x(t\eta-\eta^2)\,d\eta$$

$$= 8x\left[t\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]\Big|_0^t = 8x\left(\frac{t^3}{2}-\frac{t^3}{3}\right) = \frac{4}{3}t^3 x.$$

$u(x,t) = \sin x\cos 2t + 2xt + \frac{1}{3}t^3 x$, $-\infty < x < \infty$, $t > 0$

(2).

HW12, 2

(a) The graph y = x*(6-x) + 2 below shows the initial temperature distribution: hottest in the middle of the bar and 2 at the ends:

```
> plot(x*(6-x)+2, x = 0..6, y = 0..11);
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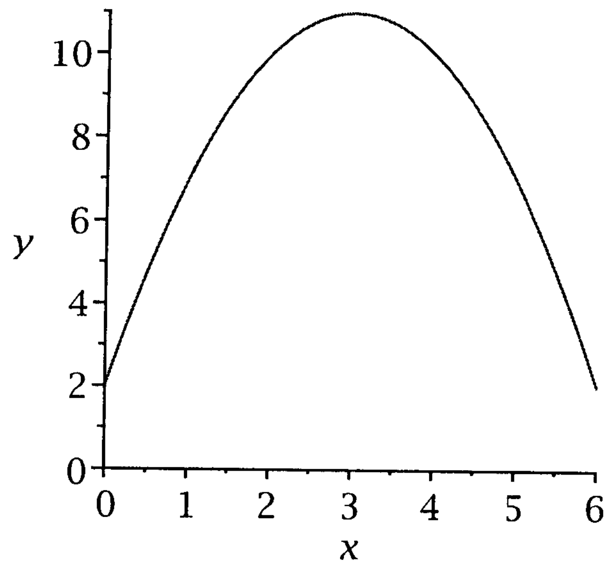


(b) Define the coefficients (c values as functions of k):

```
> c := k -> 2/6 * int(x*(6-x)*sin(((2*k+1)*Pi*x)/(2*6)), x = 0..6);
```

$$c := k \to \frac{1}{3} \int_0^6 x\,(6-x)\,\sin\left(\frac{1}{12}\,(2\,k+1)\,\pi\,x\right)\,dx \quad (1)$$

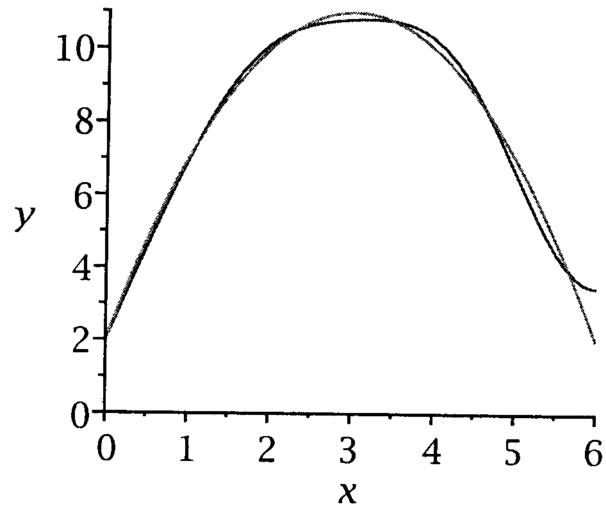
Define the nth partial sum in the series:

```
> psa := (n, t) -> 2+ sum(c(k) * exp(-(2*k+1)^2*Pi^2*4*t/(4*6^2)) * sin((2*k+1)*Pi/(2*6)*x), k = 0..n);
```

$$psa := (n, t) \to 2 + \sum_{k=0}^{n} c(k)\, e^{-\frac{1}{36}(2\,k+1)^2\,\pi^2\,t}\,\sin\left(\frac{1}{12}\,(2\,k+1)\,\pi\,x\right) \quad (2)$$

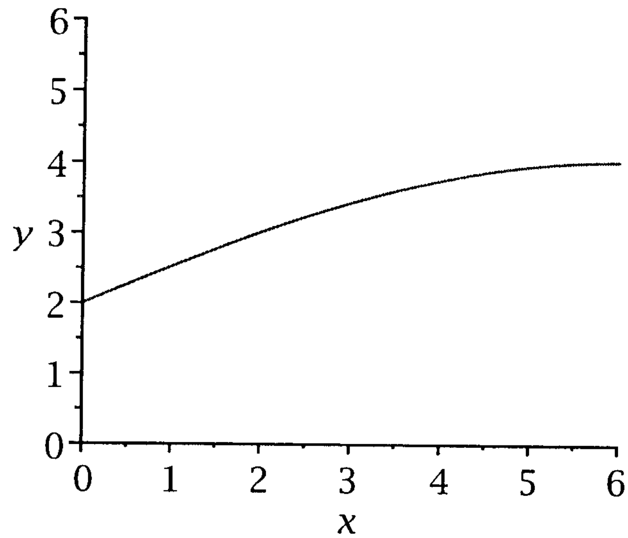
Here's a plot of the fourth partial sum for t=0 compared with the exact initial condition:

```
> plot({psa(4, 0), x*(6-x)+2}, x = 0..6, y = 0..11);
```



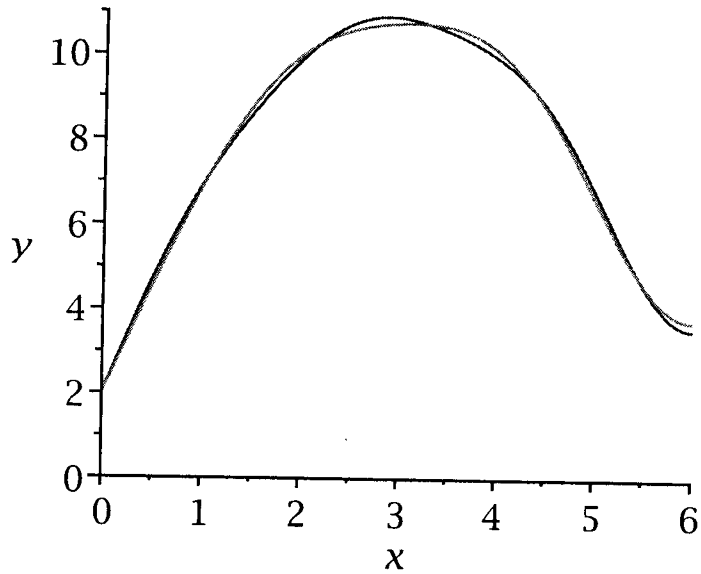
Here's a plot of the fourth partial sum for t=5: (The max temp has decreased a lot.)

```
> plot({psa(4, 5)}, x = 0..6, y = 0..6);
```

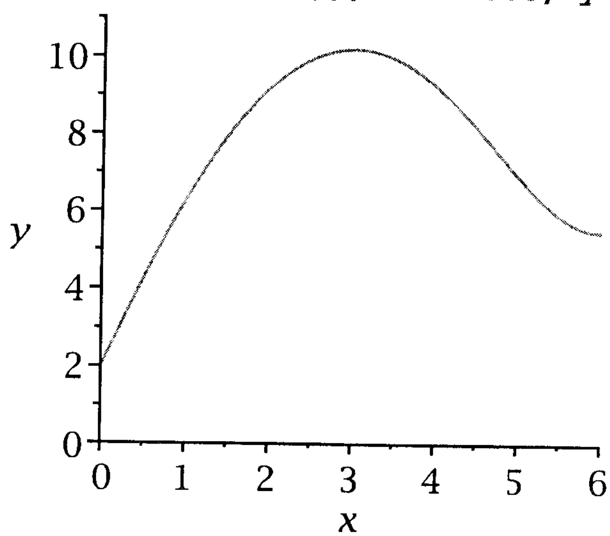


(c) The comparison with the initial condition shows good agreement away from the center and insulated boundary. At later times, the fourth partial sum is probably a reasonable approximation if including an extra term doesn't change the graph much. Compare the fourth and fifth partial sums at t = 0.01 and t = 0.1. In the latter case they are hardly distinguishable.

```
> plot({psa(4, 0.01), psa(5, 0.01)}, x = 0..6, y = 0..11);
```




```
> plot({psa(4, 0.1), psa(5, 0.1)}, x = 0..6, y = 0..11);
```



(d) At t = 3, the value at the midpoint is approximately 4.5 degrees:

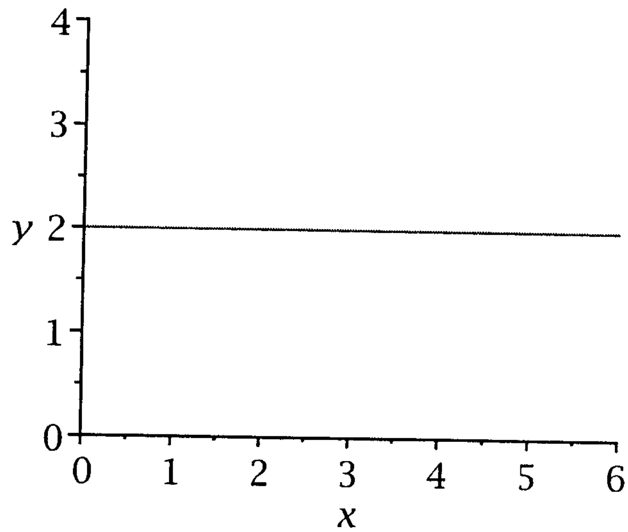
```
> evalf(subs(x = 3, psa(4, 3)));
```

4.479005422 **(3)**

(e) The temperature goes to 2 as t -> infinity. We can see this mathematically from the series representation. The amplitudes of the sine terms go to zero as t -> infinity. Physically, one end is held at 2, and the other end is insulated. Over time the bar equilibrates to the temperature of the bath at x = 0. The graph shows the approximate temperature profile at time 40.

```
> plot({psa(4, 40)}, x = 0..6, y = 0..4);
```

HW 12, 5



>

(3). $u_{tt} = c^2 u_{xx}$, $0<x<10$, $t>0$.

$u(0,t)=0$, $u(10,t)=1$, $t>0$

$u(x,0)=f(x)$, $u_t(x,0)=0$, $0<x<10$

$u(x,t)=v(x,t)+\psi(x)$; sub in $\to$.

$$\left.\begin{aligned} &v_{tt} = c^2(v_{xx}+\psi''(x)) \\ &v(0,t)+\psi(0)=0, \quad v(10,t)+\psi(10)=1 \\ &v(x,0)+\psi(x)=f(x), \quad v_t(x,0)=0 \end{aligned}\right\} \text{ if } \ldots$$

$v_{tt}=c^2 v_{xx}$

$v(0,t)=0$

$v(10,t)=0$

$v(x,0)=f(x)-\psi(x)$

$v_t(x,0)=0$ (b)

$c^2\psi''=0$

$\psi(0)=0$

$\psi(10)=1$

(a)

$\to$ Solve (a).

$\psi = ax+b$.

$\psi(0)=b=0$

$\psi(10)=a\,10=1$

$\to a=\frac{1}{10}$

$\underline{\psi(x)=\frac{1}{10}x}$

$\uparrow$ Solve (b):

See class notes.

$g(x)\to 0$

$f(x)\to f(x)-\frac{1}{10}x$ $\Rightarrow$

$L=10$

$$v(x,t)=\sum_{n=1}^{\infty}\left[c_n \sin\frac{n\pi x}{10}\cos\frac{n\pi c t}{10}+k_n \sin\frac{n\pi x}{10}\sin\frac{n\pi c t}{10}\right]$$

$$\boxed{c_n = \frac{2}{10}\int_0^{10}\left[f(x)-\frac{1}{10}x\right]\sin\frac{n\pi x}{10}\,dx}$$

$k_n = 0$

So $\boxed{u(x,t)=\frac{1}{10}x+\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10}\cos\frac{n\pi c t}{10}}$

(4). $y(x,t) = v(x,t) + \psi(x)$ sub in $\to$

$$v_{tt} = 3(v_{xx} + \psi''(x)) + 2x$$

$v(0,t) + \psi(0) = 0$,

$v(2,t) + \psi(2) = 0$,

$v(x,0) + \psi(x) = 0$,

$v_t(x,0) = 0$.

if

$3\psi''(x) + 2x = 0$.

$\psi(0) = 0$

$\psi(2) = 0$.

$v_{tt} = 3v_{xx}$

$v(0,t) = 0$

$v(2,t) = 0$

$v(x,0) = -\psi(x)$

$v_t(x,0) = 0$

$\psi''(x) = -\frac{2}{3}x$.

$\psi'(x) = -\frac{\cancel{2}}{3}\frac{x^2}{\cancel{2}} + c_1$

$\psi(x) = -\frac{1}{3}\frac{x^3}{3} + c_1 x + c_2$

$\psi(0) = c_2 = 0$.

$\psi(x) = -\frac{1}{9}x^3 + c_1 x$

$\psi(2) = -\frac{8}{9} + 2c_1 = 0$

$\to c_1 = +\frac{\cancel{8}\,4}{9\cdot\cancel{2}}$

$\psi(x) = -\frac{1}{9}x^3 + \frac{4}{9}x$

Solve

$v_{tt} = 3v_{xx}$

$v(0,t) = 0$

$v(2,t) = 0$

$v(x,0) = \frac{1}{9}x^3 - \frac{4}{9}$

$v_t(x,0) = 0$

See class notes.

$c^2 = 3$. $\quad F(x) = \frac{1}{9}x^3 - \frac{4}{9}x$

$L = 2$. $\quad g(x) = 0$

$$v(x,t) = \sum_{n=1}^{\infty} \left[c_n \sin\frac{n\pi x}{2} \cos\frac{n\pi\sqrt{3}t}{2} + \right.$$

$$\left. + k_n \sin\frac{n\pi x}{2} \sin\frac{n\pi\sqrt{3}t}{2} \right],$$

$$c_n = \frac{2}{2}\int_0^2 \left[\frac{1}{9}x^3 - \frac{4}{9}x\right] \sin\frac{n\pi x}{2}\, dx$$

$$k_n\left(\frac{n\pi\sqrt{3}}{2}\right) = \frac{2}{2}\int_0^2 0 \sin\frac{n\pi x}{2}\, dx = 0.$$

So $u(x,t) = -\frac{1}{9}x^3 + \frac{4}{9}x +$

$$+ \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2} \cos\frac{n\pi\sqrt{3}t}{2}$$

where

$$c_n = \int_0^2 \left[\frac{1}{9}x^3 - \frac{4}{9}x\right] \sin\frac{n\pi x}{2}\, dx ,$$

HW 12, 9

(5). HW 1 (9) Let $u = \exp\left(-\frac{A}{2k}x + \left(B - \frac{A^2}{4k}\right)t\right) \cdot v(x,t)$,

where $k = 1$, $A = 6$, $B = 0$

$\to$ $u = \exp(-3x - 9t)\, v(x,t)$ sub in $\to$

$v_t = v_{xx}$ (from HW 1 (9) — can also verify)

$e^{-9t} v(0,t) = 0$, $t > 0$ $\Rightarrow$ $v(0,t) = 0$

$e^{-12-9t} v(4,t) = 0$, $t > 0$ $\Rightarrow$ $v(4,t) = 0$

$e^{-3x} v(x,0) = 1$ $\Rightarrow$ $v(x,0) = e^{3x}$, $0 < x < 4$

like solved in HW 11 (1)
w/ $L = 4$, $f(x) = e^{3x}$, $k = 1$ $\to$

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{4} \exp\left(-\frac{n^2\pi^2}{16}t\right),$$

$$\boxed{c_n = \int_0^4 e^{3x} \sin\frac{n\pi x}{4}\, dx.}$$

So $$\boxed{u(x,t) = e^{-3x-9t}\left[\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{4} \exp\left(-\frac{n^2\pi^2}{16}t\right)\right]}$$