AEM II
Gross
S2008

HW 10-A soln

(1a) $[u]$ = length, $[x]$ = length, $[t]$ = time $\Rightarrow$

$\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{\text{length}}{(\text{length})^2}$, $\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\text{length}}{(\text{time})^2}$ $\Rightarrow$

$[c^2] = \frac{(\text{length})^2}{(\text{time})^2}$ so that units on right- & left-hand sides are equivalent. $\Rightarrow$

$[c]$ = velocity

(b) $[\xi] = \frac{[x]}{[L]} = \frac{\text{length}}{\text{length}} = 1$, so ξ is dimensionless

(c). $\xi = \frac{x}{L}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{L}\frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{1}{L}\frac{\partial u}{\partial \xi}\right)\frac{\partial \xi}{\partial x} = \frac{1}{L}\frac{\partial^2 u}{\partial \xi^2}\frac{1}{L} = \frac{1}{L^2}\frac{\partial^2 u}{\partial \xi^2}$$

$$\text{eq} \rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{c^2}{L^2}\frac{\partial^2 u}{\partial \xi^2} \Longleftrightarrow \frac{L^2}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2} \checkmark$$

(d) $\tau = kt$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tau}\frac{\partial u}{\partial t} = k\frac{\partial u}{\partial \tau}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \tau}\left(k\frac{\partial u}{\partial \tau}\right)\frac{\partial \tau}{\partial t} = k\frac{\partial^2 u}{\partial \tau^2}k = k^2\frac{\partial^2 u}{\partial \tau^2}$$

$$\text{eq} \rightarrow k^2\frac{L^2}{c^2}\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial \xi^2}$$

$\rightarrow$ choose $\underline{k = \frac{c}{L}}$

(1e) $[\tau] = [k][t] = \frac{[c]}{[L]}[t]$

$$= \frac{\frac{\text{length}}{\text{time}}}{\text{length}} \cdot \text{time} = 1 \checkmark$$

(2) (a) The ends of the rods are held at 0°.

(b). The initial temperature of the rod at position x is $f(x)$.

(c). $u_x = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$

$$u_{xx} = -\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$$

$$u_t = -\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}\exp\left(-\frac{n^2\pi^2 t}{L^2}\right)$$

$=$ ✓

$u(0,t) = 0$ ✓ $\quad u(L,t) = 0$ if $n \in \mathbb{Z}$.

(d) $u(x,0) = \sin\frac{n\pi x}{L} \neq f(x)$ in general

(e) The superposition satisfies the PDE & BCs because they are all linear homogeneous.

(f) $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = \sin\frac{\pi x}{L} - 3\sin\frac{4\pi x}{L}$

means $c_1 = 1$, $c_4 = -3$, & $c_n = 0$ for $n = 2,3,5,6,7,\ldots$

(g) Initially the temperature is 50° at every point on the rod.

(h) $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} = 50 \Rightarrow$ choose

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad f(x) = 50.$$

The Fourier sine series converges to 50 on $(0,L)$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} = (h_y - g_z)\mathbf{i} - (h_x - f_z)\mathbf{j} + (g_x - f_y)\mathbf{k}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ h_y - g_z & f_z - h_x & g_x - f_y \end{vmatrix} = \mathbf{i}\,(g_{xy} - f_{yy} - f_{zz} + h_{xz}) + \mathbf{j}\,(-g_{xx} + f_{yx} + h_{yz} - g_{zz}) + \mathbf{k}\,(f_{zx} - h_{xx} - h_{yy} + g_{zy})$$

$$\nabla \cdot \mathbf{F} = f_x + g_y + h_z$$

$$\nabla(\nabla \cdot \mathbf{F}) = (f_{xx} + g_{yx} + h_{zx})\mathbf{i} + (f_{xy} + g_{yy} + h_{zy})\mathbf{j} + (f_{xz} + g_{yz} + h_{zz})\mathbf{k}$$

$$\nabla^2 \mathbf{F} = (f_{xx} + f_{yy} + f_{zz})\mathbf{i} + (g_{xx} + g_{yy} + g_{zz})\mathbf{j} + (h_{xx} + h_{yy} + h_{zz})\mathbf{k}$$

$$\nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} = (g_{yx} - f_{yy} - f_{zz} + h_{zx})\mathbf{i} + (-g_{xx} + f_{xy} + h_{zy} - g_{zz})\mathbf{j} + (f_{xz} - h_{xx} - h_{yy} + g_{yz})\mathbf{k}$$

$=$ provided f, g, h have cont. second partial derivatives