HW 1-A sol$^{\underline{n}}$

AEM II
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S2008

(1) $L(y) = y'' + \sin(t+y)$, $f(t) = \sin t$

disprove:

$$L(\alpha u+\beta v) = (\alpha u+\beta v)'' + \sin(t+\alpha u+\beta v)$$

$$= \alpha u'' + \beta v'' + \sin(t+\alpha u+\beta v)$$

$$\alpha L(u) + \beta L(v) = \alpha(u'' + \sin(t+u)) + \beta(v'' + \sin(t+v))$$

$\neq$

(2) $\left[T_n'(t) + \frac{n^2\pi^2 k}{L^2} T_n(t) = B_n(t)\right] e^{\frac{n^2\pi^2 k}{L^2}t}$

$$\Rightarrow \left(e^{\frac{n^2\pi^2 k}{L^2}t} T_n(t)\right)' = B_n(t) e^{\frac{n^2\pi^2 k}{L^2}t}$$

$$\Rightarrow e^{\frac{n^2\pi^2 k}{L^2}t} T_n(t) - e^0 T_n(0) = \int_0^t B_n(\tau) e^{\frac{n^2\pi^2 k}{L^2}\tau} d\tau$$

$$\Rightarrow T_n(t) = \left[b_n + \int_0^t B_n(\tau) e^{\frac{n^2\pi^2 k}{L^2}\tau} d\tau\right] e^{-\frac{n^2\pi^2 k}{L^2}t}$$

$$= \int_0^t \exp\left(\frac{-n^2\pi^2 k}{L^2}(t-\tau)\right) B_n(\tau)\, d\tau + b_n \exp\left(-\frac{n^2\pi^2 k}{L^2}t\right)$$

✓

(3) Let $L(u) = 0$ & $L(v) = 0$.

Show $L(c_1 u + c_2 v) = 0$:

$$L(c_1 u + c_2 v) = c_1 L(u) + c_2 L(v)$$ $\left\{\text{by def of a linear operator}\right.$

$$= c_1(0) + c_2(0)$$

$$= 0$$ ✓

(4) Step 1 in the proof in (3) required L to be a linear operator.

(5) Let $L(u)=f$ & $L(v)=f$, $f\neq 0$.
Does $L(c_1u+c_2v)=f$?
For a linear operator L,

$$L(c_1u+c_2v)=c_1L(u)+c_2L(v)$$
$$=c_1f+c_2f\neq f \text{ in general.}$$

Superposition does not hold.

(6)(a) $u_1=e^{(p+iq)x}=e^{px}e^{iqx}=$

$$=e^{px}(\cos qx+i\sin qx)$$

$u_2=e^{(p-iq)x}=e^{p}e^{i(-qx)}=$

$$=e^{px}(\cos qx+i\sin(-qx))$$
$$=e^{px}(\cos qx-i\sin qx)$$

(b) $u_3=\frac{1}{2}u_1+\frac{1}{2}u_2=e^{px}\cos qx$

$u_4=\frac{1}{2i}u_1-\frac{1}{2i}u_2=e^{px}\sin qx$

$u=c_1e^{px}\cos qx+c_2e^{px}\sin qx$ ✓

(7) $-6\cancel{e^{-6x}}\,y' + \cancel{e^{-6x}}\,y'' + (1+\lambda)\cancel{e^{-6x}}\,y=0$ HW1, p3

$y = e^{rx} \Rightarrow r^2 - 6r + (1+\lambda) = 0 \rightarrow$

$$r = \frac{\cancel{6}^{3} \pm \sqrt{\cancel{36}^{9} - 4(1+\lambda)}}{\cancel{2}} = 3 \pm \sqrt{8-\lambda}$$

(I) $8-\lambda > 0 \quad (\lambda < 8)$

$$y = c_1 e^{(3+\sqrt{8-\lambda})x} + c_2 e^{(3-\sqrt{8-\lambda})x}$$

(II) $8-\lambda = 0 \quad (\lambda = 8)$

$$y = c_1 e^{3x} + c_2 e^{3x} x.$$

(III) $8-\lambda < 0 \quad (\lambda > 8)$.

$$y = e^{3x}\left(c_1 \cos(\sqrt{\lambda-8}\,x) + c_2 \sin((\sqrt{\lambda-8})\,x)\right).$$

(8). $x^2 y'' + xy' + (x^2-\nu^2)y = 0, \quad x>0, \nu \geq 0$

$x = \lambda z \quad (z = \frac{1}{\lambda}x)$.

$$\frac{dy(x(z))}{dx} = \frac{dY(z)}{dz}\frac{dz}{dx} = \frac{1}{\lambda}\frac{dY}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{1}{\lambda}\frac{dY}{dz}\right)\frac{dz}{dx} = \frac{1}{\lambda^2}\frac{d^2Y}{dz^2}.$$

$$(\lambda z)^2 \frac{1}{\lambda^2}\frac{d^2Y}{dz^2} + (\lambda z)\frac{1}{\lambda}\frac{dY}{dz} + ((\lambda z)^2 - \nu^2)Y = 0.$$

$$z^2 \frac{d^2Y}{dz^2} + z\frac{d2Y}{dz^2} + (\lambda^2 z^2 - \nu^2)Y = 0.$$

(9) $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(e^{\alpha x+\beta t} v(x,t)\right)$

$$= \beta e^{\alpha x+\beta t} v(x,t) + e^{\alpha x+\beta t}\frac{\partial v}{\partial t}.$$

$$\frac{\partial u}{\partial x} = \alpha e^{\alpha x+\beta t} v(x,t) + e^{\alpha x+\beta t}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[e^{\alpha x+\beta t}\left(\alpha v(x,t) + \frac{\partial v}{\partial x}\right)\right]$$

$$= e^{\alpha x+\beta t}\left[\alpha\left(\alpha v(x,t)+\frac{\partial v}{\partial x}\right) + \alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right]$$

$$e^{\alpha x+\beta t}\left\{\beta v + \frac{\partial v}{\partial t} = k\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) + A\left(\alpha v + \frac{\partial v}{\partial x}\right) + B v\right\}.$$

Equate like terms to drop out:

$$\beta = k\alpha^2 + A\alpha + B, \quad \boxed{\alpha = \frac{-A}{2k}}$$

$$\rightarrow \quad \beta = \frac{\cancel{k}A^2}{4k^{\cancel{2}}} - \frac{2A^2}{2\cdot 2k} + B$$

$$\boxed{\beta = \frac{-A^2}{4k} + B}$$

$$\Rightarrow \quad v_t = k v_{xx} \checkmark$$

(10) $\left(\frac{1}{x} y'\right)' + (4+\lambda) x^{-3} y = 0$

$$\left[-\frac{1}{x^2} y' + \frac{1}{x} y'' + \frac{(4+\lambda)}{x^3} y = 0 \right] x^3$$

$$x^2 y'' - x y' + (4+\lambda) y = 0 \qquad \text{Euler's eq.}$$

$$y = x^m \rightarrow$$

$$\cancel{x^2}\, m(m-1) x^{m-\cancel{2}} - \cancel{x}\, m x^{m-\cancel{1}} + (4+\lambda) x^m = 0$$

$$m^2 - m - m + (4+\lambda) = 0$$

$$m = \frac{\cancel{2} \pm \sqrt{\cancel{4} - \cancel{4}(4+\lambda)}}{\cancel{2}} = 1 \pm \sqrt{-3-\lambda}$$

(I) $-3-\lambda > 0 \quad (\lambda < -3)$:

$$y = c_1 x^{(1+\sqrt{-3-\lambda})} + c_2 x^{(1-\sqrt{-3-\lambda})}$$

(II) $-3-\lambda = 0 \quad (\lambda = -3)$.

$$y = c_1 x^{[\text{redacted}]} + c_2 x^{[\text{redacted}]} \ln x$$

(III) $-3-\lambda < 0 \quad (\lambda > -3)$: $m = 1 \pm i\sqrt{3+\lambda}$

$$y = x \left[c_1 \cos\left(\sqrt{3+\lambda} \ln x\right) + c_2 \sin\left(\sqrt{3+\lambda} \ln x\right) \right]$$