THE UNIVERSITY OF AKRON
Mathematics and Computer Science

⇑ mptii menu

Lesson 9: Functions (cont.) & First Degree Curves

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An Algebra Review in Ten Lessons

I am $\mathfrak{DPS}$

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$$a^3 a^4 = a^7 \quad (ab)^{10} = a^{10} b^{10}$$

$$-(ab - (3ab - 4)) = 2ab - 4$$

$$(ab)^3 (a^{-1} + b^{-1}) = (ab)^2 (a + b)$$

$$(a - b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$$

$$2x^2 - 3x - 2 = (2x + 1)(x - 2)$$

$$\tfrac{1}{2} x + 13 = 0 \implies x = -26$$

$$\mathrm{G} = \{ (x, y) \mid y = f(x) \}$$

$$f(x) = mx + b$$

$$y = \sin x$$

Last Revision Date: 2/2/2000

Lesson 9: Functions (cont.) & First Degree Curves

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9. Functions (cont.) & First Degree Curves

9.1. Functions Revisited

- **Is y a function of x?**

In everyday conversational speech we say that "something is a function of something else." For example, "success is a function of effort," "postage on a letter is a function of its weight," "the diameter of a tree is a function of its age," and so on. These are statements assert that a measure of one quantity, in some way, depends on the measure of another quantity. However, in conversational speech, the exact way in which one quantity depends on the other is not specified; indeed, the exact relationship may be unknown. These are usually statements of "feelings" or "relationships."

In mathematics, we have the same type statements, "y is a function of x," but here, the meaning is precise.

We say that y is a function of x if we can find a function $f(x)$ such that

$$y = f(x).$$

There is no particular significance to the letters x and y; likewise, we can say that x is a function of y provided there is some function $g(y)$ such that

$$x = g(y).$$

What does it mean for w to be a function of s? It means that the variable w is expressible *in terms of* s:

$$w = h(s),$$

for some function $h(s)$.

Illustration 1. Here are some examples of functions of different variables.

(a) $y = 4x^3 - 3x + 1$ defines y as a function of x.
(b) $x = 5y^4 - 8y + 12$ defines x as a function of y.
(c) $w = 2t + 1$ defines w as a function of t.
(d) $t = \frac{1}{2}(w - 1)$ defines t as a function of w.

(e) $A = \pi r^2$ defines A, area, as a function of r, the radius of a circle.
(f) $C = 2\pi r$ defines circumference of a circle, C, as a function of r, the radius.
(g) $r = \dfrac{C}{2\pi}$ defines radius, r, as a function of C.

■

EXAMPLE 9.1. Given $y = 3x - 1$. Write x as a function of y.

EXERCISE 9.1. Respond to each of the following.
(a) Given $y = 5x + 1$, write x as a function of y.
(b) Given $4x - 2y = 1$, write y as a function of x.
(c) Given $4x - 2y = 1$, write x as a function of y.
(d) Given $2w + 3s^3 = 3$, write s as a function of w.
(e) Given $2w + 3s^3 = 3$, write w as a function of s.

In applications, you must create your own function of the appropriate type.

EXERCISE 9.2. Suppose we have a cube and the total surface area is known. The problem is to find the length of the common side of the cube. *Your assignment*: Write the length, x, of the side of a cube as a function of the surface area, S; i.e., write $x = f(S)$. Use your formula to compute the length of the common side of a cube having a total surface area of 24 square units. ■

Exercise 9.3. Write the radius, r, of a sphere as a function of its volume, V.

Exercise 9.4. Write the radius, r, of a sphere as a function of its surface area, S.

Exercise 9.5. Solve each of the following: (a) A circle has a radius of $r = 3$ units. Find the circumference, C, of the circle; (b) Another circle has a circumference of $C = 20\pi$, find the radius, r, of this circle. (*Reference*: Part (f) and Part (g) of Illustration 1.)

- **Graphing**

As you know, any equation in two variables, x and y, can be represented as a *curve* in the plane. This curve is called the *graph* of the equation.

We symbolically denote an equation by

$$F(x, y) = c \tag{1}$$

where the left-hand side, $F(x, y)$, represents some expression in two variables, and the right-hand side c, is some constant. The graph of (1) is the set of all points (x, y) in the plane that *satisfy the equation*. (Sometimes we say that the graph is the *locus* of all points that satisfy the given equation.)

When we identify the points on the graph of an equation, it is typically a curve in the plane. The graph, then, is a visual and geometric manifestation of the equation (1).

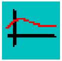

Figure 1

For example, consider the equation $x^{2/3} + y^{2/3} = 2$. The point $(1, 1)$ satisfies the equation since $1^{2/3} + 1^{2/3} = 2$; in fact, all the points $(1, 1)$, $(-1, 1)$, $(1, -1)$ and $(-1, -1)$ satisfy the equation. These, though, are only four of *infinitely many* points. If we identify enough points we can get a "feel" for the shape of the curve and are thus able to draw it in. Figure 1 represents the graph of this curve.

We shall not go into the technicalities of graphing; in the age of the graphing calculator, many of the these age-old techniques seem antiquated. Rather, we shall concentrate on study in this lesson, on the properties of the straight line. In LESSON 10 we shall look at additional curves: the parabola, the circle, and an introduction to the circular (or trig) functions.

Some curves are the graphs of functions, others are not. The first order of business is to classify curves as *functional curves* or not. In the next section, we develop a basic, yet important, criteria for doing exactly this.

• The Vertical Line Test

Curves that are graphs of functions are particularly important. Let $y = f(x)$ be a function. (In this case, we say that y *is a function of* x.) We may look upon $y = f(x)$ as an equation as discussed above. (Rewrite $y = f(x)$ as $y - f(x) = 0$, this makes is look like (1); however, it is usually inconvenient to write it this way.) The fact that y is a function of x means that for any choice of $x = x_0$ in the domain of f, there corresponds a unique $y = y_0$. This (x_0, y_0) pair will satisfy the equation because that is the definition of a function: If y_0 corresponds to x_0 with respect to the function f, then $y_0 = f(x_0)$.

Now for the problem: Suppose you have a curve $\mathcal{C}$ drawn in the xy-plane. How can we tell whether this curve $\mathcal{C}$ represents y as a function of x?

There is a simple graphical test.

> *Vertical Line Test*:
> A curve $\mathcal{C}$ in the xy-plane defines y as a function of x if it is true that every vertical line intersects the curve at *no more than* one point.

Important. The x-axis is assumed to be the *horizontal axis*, and so the meaning of *vertical* is perpendicular to the x-axis.

Exercise 9.6. Taking the definition of function into consideration, the orientation of the axes (x-axis is horizontal), and the geometry of the graph of a curve, justify in your own mind the *Vertical Line Test*.

The equation $y = 2x + 1$ defines y as a function x—here, $f(x) = 2x + 1$. Usually, we like the independent variable to be on the x-axis (the horizontal axis); sometimes, however, things do not work out as planned. The equation $x = y^5 - y$ defines x as a function of y.

The concept of function is independent of the letters chosen to express the relationship; here, in this instance, $x = y^2 - y$ does define x as a function of y—for each value of y there corresponds only one x. Now the independent variable y is one the *vertical axis* and the dependent variable is on the *horizontal axis*. I hope this does not disturb you psychologically too much.

Exercise 9.7. When x is a function of y, $x = g(y)$, and we graph the equation in the xy-plane, what are the distinguishing characteristics of the curve that determine it to be the graph of a function on y?

Given a curve in the plane we can discern whether it is a graph of a function using the Vertical Line Test and its variant, the Horizontal Line Test.

Quiz. Answer each of the following questions. Passing is 100%.

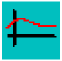
Figure 2

1. Does this curve define y a function of x?

(a) Yes (b) No

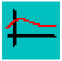
Figure 3

2. Does this curve define y a function of x?

(a) Yes (b) No

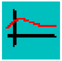
Figure 4

3. Does this curve define y a function of x?

(a) Yes (b) No

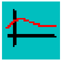
Figure 5

4. Does this curve define x a function of y?

(a) Yes (b) No

End Quiz.

9.2. Lines

Two points determine a line. We all know that. The problem is how to express a line mathematically in a way we can manipulate it algebraically.

Let's dispatch two simple cases first.

▷ *Vertical Lines*: Let ℓ be a vertical line. Characteristic of the points on ℓ is that they all have the same *first coordinate*. Let a be the x-intercept of the line ℓ. Then an equation that describes the line ℓ is given by

$$\boxed{x = a.} \tag{2}$$

Thus, ℓ is the set of all points that satisfy the above equation; i.e., all points having an x-coordinate of a.

For example, the equation $x = 3$ is the vertical line that crosses the x-axis at 3.

▷ *Horizontal Lines*: Let ℓ be an horizontal line. Characteristic of the points of ℓ is that they all have the same *second coordinate*. Let b be the y-intercept of the line ℓ, then the equation that describes the line ℓ is given by

$$\boxed{y = b.} \tag{3}$$

The line ℓ consists of all points that satisfy the equation; i.e., all points whose second coordinate is b.

For example, $y = -4$ is the horizontal line crossing the y-axis at -4.

For nonvertical lines, the analysis is not quite so simple. We begin by a discussion of the *slope of a line.*

- **The Slope of a Line**

Let ℓ be a nonvertical line. Choose any two distinct points on ℓ; call them $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then P and Q are *not* vertically oriented.

Question. What does it mean for P and Q *not* to be vertically oriented?

(a) $x_1 = x_2$ (b) $x_1 \neq x_2$ (c) $y_1 = y_2$ (d) $y_1 \neq y_2$

Slope of a Line:
Let ℓ be a nonvertical line and let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on this line. The *slope*, m, of the line ℓ is given defined to be

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{4}$$

Comments: The value of the slope, m, does not depend on the two points chosen to compute it. Indeed, if $P(x_1, y_1)$ and $Q(x_2, y_2)$ are chosen from the line, and if $P'(x'_1, y'_1)$ and $Q'(x'_2, y'_2)$ are another pair of points from the line, then by the principle of similar triangles FIGURE 6, we have

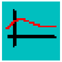

Figure 6

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y'_2 - y'_1}{x'_2 - x'_1}. \tag{5}$$

We compute the slope by taking the difference in the ordinates of the two points and dividing by the difference in their abscissa—begin sure to subtract in the same order. ■

EXAMPLE 9.2. Calcuate the slope of the line that passes through $P(-1, 3)$ and $Q(4, 1)$.

EXERCISE 9.8. Find the slope of the line passing through each of the given pair of points.

(a) $P(3, 1)$ and $Q(4, 5)$ (b) $P(-2, -3)$ and $Q(0, 2)$
(c) $P(-\frac{1}{2}, \frac{2}{3})$ and $Q(-3, 2)$ (d) $P(2, 1)$ and the origin

• The Two-Point Form

Now let's turn to the problem of creating an equation that describes a line.

Given two points $P(x_1, y_1)$ and $Q(y_1, y_2)$. These two points determine a unique line, namely the line passing through P and Q; call this line

ℓ. The slope of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{6}$$

As was remarked earlier, the slope calculation does not depend on the particular points on the line ℓ. If we take a generic point $R(x, y)$ from the line it should be true that

$$m = \frac{y - y_1}{x - x_1}. \tag{7}$$

That is, m is also determined by the points R and P.

Now equate equations (6) and (7) to obtain

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This last equation, called the two-point form of the equation of a line, characterizes the points on the line ℓ.

Two-Point Form:
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two nonvertical points in the plane. Then the equation of the line passing through these two points is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{8}$$

EXAMPLE 9.3. Find the equation of the line passing through the two points $P(-2, 4)$ and $Q(6, 9)$.

Having obtained the equation of a line, we can start to extract *information* from it.

EXAMPLE 9.4. For the line obtained in EXAMPLE 9.3, find the y-intercept, the x-intercept, and the points on the graph corresponding to $x = -3$ and $x = 6$.

EXERCISE 9.9. Find the equation of the line passing through the two points $P(-4,-1)$ and $Q(5,1)$.

EXERCISE 9.10. Given the equation developed in the EXERCISE 9.9, find the y-intercept and the x-intercept.

EXERCISE 9.11. Show that the equation for the line that crosses the x-axis at $x = 2$ and the y-axis at $y = 5$ can be written in the form

$$\frac{x}{2} + \frac{y}{5} = 1.$$

EXERCISE 9.12. (The Two-Intercept Form) Suppose a line crosses the x-axis at $x = a$ and the y-axis at $y = b$. Show that the equation for this line can be written as

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{9}$$

The two-intercept form can be a very quick way of writing down an equation for a line if you know both intercepts.

EXERCISE 9.13. Write the equation of the line that crosses the x-axis at $x = 4$ and the y-axis at $y = -3$.

• The Point-Slope Form

The two-point form was introduced above as a transitional form; i.e., you use it to establish an initial equation and then you manipulate the equation to put it into a more useful form. The point-slope form is the same way.

The Point-Slope Form:
Suppose it is known that a line ℓ has slope m and passes through a point $P(x_1, y_1)$. The equation of the line ℓ is given by

$$y - y_1 = m(x - x_1) \tag{10}$$

EXAMPLE 9.5. Find the equation of the line with a slope of $m = 5$ and passes through the point $P(1, 2)$. Leave the answer in the form of y as a function of x.

EXERCISE 9.14. Find the equation of the line that passes through the point $P(3, -6)$ and having slope $m = -2$. Leave your answer in the form of y as a function of x. Find another point on the line and using these two points, make a sketch of the line.

EXERCISE 9.15. Find the equation of the line with a slope of $m = -\frac{1}{2}$ that passes through the point $P(-2, -9)$. Leave your answer in the form of x as a function of y.

- **The Slope-Intercept Form**

The slope-intercept form is one of the final forms of an equation of a line, as opposed to a transitory form. The point-slope form is $y - y_1 =$

$m(x - x_1)$. As we did in the examples and exercises in the previous paragraphs, we wrote this equation in the form of a function of x:

$$\begin{aligned} y &= y_1 + m(x - x_1) \\ &= y_1 + mx - mx_1 \\ &= mx + (y_1 - mx_1) \\ &= mx + b \qquad \triangleleft \text{ put } b = y_1 - mx_1 \end{aligned}$$

The equation $y = mx + b$ is called the slope-intercept form of the equation of a line. Of course, m is the slope of the line. But what is the interpretation of the number b? The y-intercept of a line is obtained by putting $x = 0$ and solving for y. If we do that here we get

$$y = mx + b\,\Big|_{x=0} = b$$

Thus, $y = b$ is the y-intercept of the line.

Let's record this in shadow box form.

Slope-Intercept Form:
The equation of the line ℓ having a slope of m and a y-intercept of b given by

$$y = mx + b \tag{11}$$

EXERCISE 9.16. Find the equation of the line that has slope 6 and crosses the y-axis at -1.

EXERCISE 9.17. A line crosses the y-axis at $y = 6$ and has a slope of $m = -2$. Find the equation of the line and leave your answer in the form of a function of x.

EXERCISE 9.18. Find the equation of the line that passes through the two points $P(3, 1)$ and $Q(4, 5)$. Leave your answer in the infamous slope-intercept form, $y = mx + b$. (*Reference*: two-point form.)

EXERCISE 9.19. A line has slope $m = \frac{1}{2}$ and passes through the point $P(-2, -5)$, find the equation of this line and leave your answer in the slope-intercept form; i.e., leave your answer in the form of y as a function of x. (*Reference*: The point-slope form.)

Quiz. Copy down the equation of the line described in EXERCISE 9.19 and use it to answer each of the following questions. Passing is 100%.

1. What is the y-intercept of this line?

(a) -8 (b) -4 (c) 4 (d) 8

2. Which of the following is the x-intercept of the line?

(a) -8 (b) -4 (c) 4 (d) 8

3. Is the point $(2, -1)$ on this line?

(a) Yes (b) No

4. Write the equation for the line with x as a function of y:

(a) $x = 2y + 8$ (b) $x = 2y - 8$ (c) $x = \frac{1}{2}y + 4$ (d) $x = \frac{1}{2}y - 4$

5. Use the results of **Question #4** to obtain the abscissa (the x-coordinate) of the line when the line has an ordinate (a y-coordinate) of 10.

(a) 1 (b) 10 (c) 14 (d) 28

EndQuiz.

This all seems all very simple. Given certain information, it is easy to compute the equation of the line.

EXERCISE 9.20. Consider the two lines $y = 3x - 1$ and $y = 1 - x$.

(a) Compute the slopes of each of these two lines

(b) State the y-intercepts of each line.

EXERCISE 9.21. The two lines in EXERCISE 9.20 intersect each other. Find the point of intersection.

- **The General Form**

All the forms for the equation of a line can ultimately be written as

$$Ax + By + C = 0 \tag{12}$$

This is called the **General Form** for the equation of a line.

For example, in Exercise 9.19 we obtained the equation for a line as $y = \frac{1}{2}x - 4$, written in the slope-intercept form. After a few algebraic steps, we obtain $x - 2y - 8 = 0$, which is in the same form as equation (12).

The **General Form**, like many of the forms, is not unique. For example, we can write $x - 2y - 8 = 0$ as $\frac{1}{2}x - y - 4 = 0$, or as $3x - 6y - 24 = 0$. *Usually though*, we are true to our algebraic roots and remove any common factors (so $3x - 6y - 24 = 0$ is not considered "good form"); and if the coefficients are all *rational numbers*, we usually *clear fractions* and write the equation with *integer coefficients* (so $\frac{1}{2}x - y - 4 = 0$ is not a preferred form).

We can also put the constant term to the other side of the equation; thus, $x - 2y = 8$ would be considered in the **General Form** of the equation of a line.

Example 9.6. Calculate the slope of the line $2x - 3y = 5$.

EXERCISE 9.22. Find the slope and y-intercept of each of the following equations.
(a) $5x + 3y - 1 = 0$ (b) $-3x + 12y + 13 = 0$ (c) $x - y = 0$

EXERCISE 9.23. Sketch the graph of each line in EXERCISE 9.22 by plotting the y-intercept and by finding and plotting an additional point.

EXERCISE 9.24. Find the equation of the line that passes through the points $(-4, 1)$ and $(5, 1)$. Leave your answer in the **General Form**.

Recall, the y-intercept is obtained by putting $x = 0$ and solving for y, and the x-intercept is obtained by putting $y = 0$ and solving for x. Use these simple criteria to solve the next problem.

EXERCISE 9.25. Find the x- and y-intercepts of each of the lines, and sketch their graphs by plotting the intercepts and drawing a line through them.
(a) $3x + 4y = 24$ (b) $5x - 2y = 10$ (c) $x - 2y = 1$

EXERCISE 9.26. Find the points of intersection between the following pairs of lines. (**Warning:** In one of the parts below, the lines do *not* intersect.) Leave your answer should be a point in the plane: $P(a, b)$.

(a) $x - y = 1$ and $x + y = 1$ (b) $x - 3y = 1$ and $2x - y = 1$
(c) $4x - 2y = 7$ and $y = 2x + 1$ (d) $6x + y = 3$ and $x + y = 2$

• Parallel & Perpendicular Lines

The last topic under consideration in this lesson is to develop criteria for determining whether two lines are parallel or perpendicular.

▷ *Parallel Lines*: Two lines ℓ_1 and ℓ_2 are parallel if and only if they do not intersect. Suppose the equations of these lines are $y = m_1 x + b_1$ and $y = m_2 x + b_2$, respectively. When we try to find the point of intersection we equate

$$m_1 x + b_1 = m_2 x + b_2$$

and try to solve for x:

$$(m_1 - m_2)x = b_2 - b_1.$$

We can divide by $m_1 - m_2$ to get the solution for x, *provided* $m_1 - m_2 \neq 0$. It is the singular case of $m_1 - m_2 = 0$ in which we cannot solve for x; this is the case in which the lines are parallel.

Parallel Lines:
Two lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 respectively, are parallel if and only if

$$m_1 = m_2 \tag{13}$$

that is, if and only if they have the same slope.

When the lines are written as functions of x, i.e., in the slope-intercept form, it is trivial to see whether the lines are parallel. For example, $y = 2x - 5$ and $y = 2x + 12$ are parallel because the slope of each line is $m = 2$. The two lines $y = 3x - 4$ and $y = 5x + 1$ are *not* parallel because the first line has slope $m_1 = 3$ and the second line has slope $m_2 = 5$.

When the lines are in general form some slight effort is needed to determine whether two lines are parallel—simply put the equations in the infamous slope-intercept form, at which point you can make an easy determination.

Quiz. Which of the following pairs of lines are parallel. Passing is 100%.

1. Are the lines $x + 2y + 3 = 0$ and $3x + 6y + 1 = 0$ parallel?

(a) Yes (b) No

2. Are the lines $3y - 2x = 3$ and $12y - x = 6$ parallel?

(a) Yes (b) No

3. Are the lines $5x - 2y = 1$ and $4y - 10x = 2$ parallel?

(a) Yes (b) No

EndQuiz.

Exercise 9.27. Find the equation of the line that is parallel to the line $y = 2x - 1$ but passes through the point $P(2, 7)$.

Exercise 9.28. Find the equation of the line that passes through the point $P(-3, -1)$ and is parallel to the line $5x + 4y = 12$.

▷ *Perpendicular Lines*: Two lines ℓ_1 and ℓ_2 are **perpendicular** or **orthogonal** if they intersect and at the point of intersection the angle between the two lines is 90°. We state the following criteria with proof.

Perpendicular Lines:
Let ℓ_1 be a line with slope $m_1 \neq 0$ and ℓ_2 be a line with slope $m_2 \neq 0$, then ℓ_1 is perpendicular to ℓ_2 ($\ell_1 \perp \ell_2$) if and only if

$$m1 \cdot m_2 = -1 \text{ or } m_2 = -\frac{1}{m_2}, \tag{14}$$

that is, if and only if the slope of one line is the *negative reciprocal* of the other.

Exercise 9.29. Think about the cases excluded explicitly in the criteria for perpendicular lines. When are lines that fall into the exceptional case perpendicular?

Quiz. Which of the following pairs of lines are perpendicular to each other. Passing is 100%.

1. Are the lines $y = 2x + 1$ and $y = -\frac{1}{2}x - 3$ perpendicular?

(a) Yes (b) No

2. Are the lines $x - 3y + 3 = 0$ and $3x + y - 2 = 0$ perpendicular?

(a) Yes (b) No

3. Are the lines $3x - 6y = 1$ and $9x + 3y = 2$ perpendicular?

(a) Yes (b) No

EndQuiz.

Exercise 9.30. Find the equation of the line that passes through the point $P(3, 4)$ and is perpendicular to the line $y = 5x + 1$.

Exercise 9.31. Find the equation of the line that passes through the point $P(-5, 3)$ and is perpendicular to the line $6x + 2y = 1$.

We have now developed enough tools to successfully tackle the following problem.

EXERCISE 9.32. Find the distance the point $P(-1, -1)$ is away from the line $y = 2x + 4$. Answer this question by first mapping out a strategy, then carrying out your strategy. Parts (a) and (b) correspond to each of these two steps.

(a) Plot the point and the line, then map out in a series of steps how you plan to solve this problem.

(b) Now carry out your game plan.

We have come to the end of LESSON 9. Lesson 10 continues the discussion of some common, yet important curves: parabolas, circles, and trig functions.

Solutions to Exercises

9.1. *Solutions*:

(a) Given $y = 5x + 1$, write x as a function of y.

$$\boxed{\textit{Answer}\colon x = \tfrac{1}{5}(y-1)}$$

(b) Given $4x - 2y = 1$, write y as a function of x.

$$\boxed{\textit{Answer}\colon y = \tfrac{1}{2}(4x-1)}$$

(c) Given $4x - 2y = 1$, write x as a function of y.

$$\boxed{\textit{Answer}\colon x = \tfrac{1}{4}(1+2y)}$$

(d) Given $2w + 3s^3 = 3$, write s as a function of w.

$$\boxed{\textit{Answer}\colon s = \sqrt[3]{\frac{3-2w}{3}}}$$

(e) Given $2w + 3s^3 = 3$, write w as a function of s.

$$\boxed{\textit{Answer}\text{: } w = \frac{3}{2}(1 - s^3)}$$

Exercise 9.1. ■

9.2. *Solution*: As the first step, find surface area, S, as a function of x. Why? Because its easy!

$$S = 6x^2$$

But we don't want this. We want x as a function of S. No problem:

$$\boxed{x = \sqrt{\frac{S}{6}}}$$

Now, given that we have a cube with surface area of $S = 24$, we can now easily compute the length of the common side:

$$x = \sqrt{\frac{S}{6}}\,\Bigg|_{S=24} = \sqrt{\frac{24}{6}} = \sqrt{4} = 2$$

(Here, the vertical bar, $|$, is the so-called, evalutional notation—a standard way of conveying the idea that we are substituting a particular value into a formula.) Thus,

$$\boxed{S = 24 \implies x = 2}$$

Exercise 9.2. ■

9.3. *Solution*: It is well know that $V = \frac{4}{3}\pi r^3$. This represents V as a function of r. We want r as a function of V. Thus,

$$\boxed{r = \sqrt[3]{\frac{3V}{4\pi}}}$$

Given the volume, V, this formula lends itself to the easy calculation of the radius.

Exercise 9.3. ■

9.4. *Solution*: It is well known that $S = 4\pi r^2$. This represents S as a function of r. We want r as a function of S. Thus,

$$\boxed{r = \sqrt{\frac{S}{4\pi}}}$$

Given the surface area, S, this formula lends itself to the easy calculation of the radius. Exercise 9.4. ■

9.5. *Solution to* (a) From Part (f), we have $C = 2\pi r$. This defines C as a function of r; what this means is that this equation is designed to give back certain information, C, given that you know the value of r—and we do.

$$C = 2\pi r\big|_{r=3} = 2\pi(3) = 6\pi$$

A circle of radius $r = 3$ has a circumference of $\boxed{C = 6\pi.}$

Solution to (b) From Part (g), we have $r = \dfrac{C}{2\pi}$. This defines r as a function of C.

Now we are given that $C = 20\pi$; therefore,

$$r = \frac{C}{2\pi}\bigg|_{C=20\pi} = \frac{20\pi}{2\pi} = 10$$

The radius of a circle with circumference $C = 20\pi$ is $\boxed{r = 10}$

Exercise 9.5. ■

9.6. I cannot justify it in your *own* mind, only in mine.

Exercise 9.6. ■

9.7. For each y we can have at most one x; therefore, if we draw a horizontal line at position y on the y-axis, the line should intersect the graph of $x = g(y)$ in at most one location.

This is the horizontal analog to the Vertical Line Test. We can formulate this into a test procedure.

> *Horizontal Line Test*:
> A curve $\mathcal{C}$ in the xy-plane defines x as a function of y if it is true that every horizontal line intersects the curve at *no more than* one point.

Exercise 9.7. ■

9.8. *Solutions*:

(a) $P(3,1)$ and $Q(4,5)$

$$m = \frac{5-1}{4-3} = \boxed{4}$$

(b) $P(-2,-3)$ and $Q(0,2)$

$$m = \frac{2-(-3)}{0-(-2)} = \boxed{\frac{5}{2}}$$

(c) $P(-\frac{1}{2}, \frac{2}{3})$ and $Q(-3,2)$

$$m = \frac{2-\frac{2}{3}}{-3-(-\frac{1}{2})} = \frac{\frac{4}{3}}{-\frac{5}{2}} = \boxed{-\frac{8}{15}}$$

(d) $P(2,1)$ and the origin, i.e., take $Q(0,0)$.

$$m = \frac{1-0}{2-0} = \boxed{\frac{1}{2}}$$

Exercise 9.8. ■

9.9. *Solution*: Substituting into equation (8), to obtain

$$\text{Given: } P(-4,-1) \text{ and } Q(5,1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - (-1) = \frac{1 - (-1)}{5 - (-4)}(x - (-4))$$

$$y + 1 = \frac{2}{9}(x + 4)$$

or,

$$\boxed{y = \frac{2}{9}(x + 4) - 1}$$

Exercise 9.9. ■

9.10. *Solution*: *The y-intercept* is found by putting $x = 0$ and solving for y:

$$y = \frac{2}{9}(x+4) - 1\Big|_{x=0} = \frac{2}{9}(4) - 1 = -\frac{1}{9}$$

The y-intercept is $\boxed{y = -\frac{1}{9}}$

The x-intercept is found by putting $y = 0$ and solving for x:

$$\begin{aligned} \frac{2}{9}(x+4) - 1 &= y = 0 \\ \frac{2}{9}(x+4) &= 1 \\ x + 4 &= \frac{9}{2} \\ x &= \frac{9}{2} - 4 = \frac{1}{2} \end{aligned}$$

The x-intercept is $\boxed{x = \frac{1}{2}}$

Thus, this line crosses the x-axis at $x = -1/9$ and crosses the y-axis at $y = 1/2$. It is now a trivial matter to draw the graph of the line.

Exercise 9.10. ■

9.11. *Solution*: The x-intercept is $x = 2$ and the y-intercept is $y = 5$; this means the line passes through the points $P(2, 0)$ and $Q(0,)$. Therefore,

$$\text{Given: } P(2,0) \text{ and } Q(0,5)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y = -\frac{5}{2}(x - 2) \quad \triangleleft \text{ substitute in}$$

Now we try to put the last equation in the prescribed form.

$$y = -\frac{5}{2}(x - 2) \quad \triangleleft \text{ starting point}$$
$$5x + 2y = 10 \quad \triangleleft \text{ clear fractions and expand}$$
$$\frac{5x}{10} + \frac{2y}{10} = 1 \quad \triangleleft \text{ divide by 10}$$
$$\frac{x}{2} + \frac{y}{5} = 1 \quad \triangleleft \text{ the required equation!}$$

Exercise 9.11. ■

9.12. *Solution*: The solution follows along the same lines as the previous exercise.

Given: $P(a,0)$ and $Q(0,b)$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \triangleleft \text{ two-point form}$$

$$y - 0 = \frac{b - 0}{0 - a}(x - a) \qquad \triangleleft \text{ substitute in}$$

$$y = -\frac{b}{a}(x - a) \qquad \triangleleft \text{ simplify}$$

Now we try to put the last equation in the prescribed form.

$$bx + ay = ab \qquad \triangleleft \text{ clear fractions and expand}$$

$$\frac{bx}{ab} + \frac{ay}{ab} = 1 \qquad \triangleleft \text{ divide by } ab$$

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \triangleleft \text{ the required equation!}$$

Exercise 9.12. ■

9.13. *Solution*: We use the two-intercept form:

$$\frac{x}{4} + \frac{y}{-3} = 1$$

$$\boxed{\frac{x}{4} - \frac{y}{3} = 1}$$

That was quick! Exercise 9.13. ■

9.14. *Solution*: We use equation (10):

$$\text{Given: } m = -2 \text{ and } P(3, -6)$$

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ equation (10)} \\ y - (-6) &= -2(x - 3) && \triangleleft \text{ substitute} \\ y + 6 &= -2(x - 3) && \triangleleft \text{ a rewrite} \\ y &= -6 - 2(x - 3) && \triangleleft \text{ write as a function of } x \\ y &= -2x && \triangleleft \text{ combine} \end{aligned}$$

Presentation of Answer: $\boxed{y = -2x}$

The choice of a second point is entirely up to you. I'll take $x = 1$.

$$x = 1 \implies y = -2x\big|_{x=1} = -2$$

Thus the point $Q(1, -2)$ lies on the equation. Now plot the given point $P(3, -6)$ and the computed point $Q(1, -2)$, and draw a straight line through these two points. Exercise 9.14. ■

9.15. *Solution*: We use equation (10):

$$\text{Given: } m = -\frac{1}{2} \text{ and } P(-2, -9)$$

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \triangleleft \text{ equation (10)} \\
y - (-9) &= -\frac{1}{2}(x - (-2)) && \triangleleft \text{ substitute} \\
y + 9 &= -\frac{1}{2}(x + 2) && \triangleleft \text{ a rewrite} \\
y &= -9 - \frac{1}{2}(x + 2) && \triangleleft \text{ write as a function of } x \\
y &= -\frac{1}{2}x - 10 && \triangleleft \text{ combine}
\end{aligned}
$$

Thus, $y = -\frac{1}{2}x - 10$ is the equation of the target line written as a function of x. It was requested that the answer be written in the form of x as a function of y.

$$y = -\frac{1}{2}x - 10 \implies x = -2(y + 10)$$

Presentation of Answer: $\boxed{x = -2(y + 10)}$

Exercise 9.15. ■

9.16. *Solution*: We are given the slope, $m = 6$, and the y-intercept; this is a job for the slope-intercept form of the equation of a line.

$$\text{Given: } m = 6 \text{ and } b = -1$$

$$y = mx + b \quad \triangleleft \text{ slope-intercept form}$$

$$y = 6x - 1 \quad \triangleleft \text{ substitute and done!}$$

Presentation of Answer: $\boxed{y = 6x - 1}$

Comments: Be neat and organized in your presentation. It is important to acquire a mathematical literacy. It is more difficult for me to type out a neat, well-organized presentation than it is for you to write one out. 𝔇𝔓𝔖

Exercise 9.16. ■

9.17. *Solution*: This is a job for the slope-intercept form of the equation of a line.

$$\text{Given: } m = -2 \text{ and } b = 6$$

$$y = mx + b \qquad \triangleleft \text{ slope-intercept form}$$

$$y = -2x + 6 \qquad \triangleleft \text{ substitute and done!}$$

Presentation of Answer: $\boxed{y = -2x + 6}$

Exercise 9.17. ■

9.18. *Solution*: We are given two points. We could use the two-point form, or we could (1) compute the slope, then (2) use the point-slope form. I'll take the latter tack:

1. Compute the slope.

   Given: $P(3,1)$ and $Q(4,5)$,

$$m = \frac{5-1}{4-3} = 4$$

2. Now use the point-slope form.

   Given/Known: $P(3,1)$ and $m = 4$,

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\ y - 1 &= 4(x-3) && \triangleleft \text{ substitute} \\ y &= 1 + 4x - 12 && \triangleleft \text{ expand} \\ y &= 4x - 11 && \triangleleft \text{ combine} \end{aligned}$$

Presentation of Answer: $\boxed{y = 4x - 11}$

Exercise 9.18. ■

9.19. *Solution*: Given $P(-2,-5)$ and $m = \frac{1}{2}$,

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\
y - (-5) &= \tfrac{1}{2}(x - (-2)) && \triangleleft \text{ substitute} \\
y + 5 &= \tfrac{1}{2}(x + 2) && \\
y &= -5 + \tfrac{1}{2}x + 1 && \triangleleft \text{ expand} \\
y &= \tfrac{1}{2}x - 4 && \triangleleft \text{ combine}
\end{aligned}
$$

Presentation of Answer: $\boxed{y = \frac{1}{2}x - 4}$

Exercise 9.19. ■

9.20. *Solution*:

(a) Compute the slopes of each of these two lines.

- The slope of the line $y = 3x - 1$ is $m = 3$, the coefficient of the x term.
- The slope of the line $y = 1 - x$ is $m = -1$, the coefficient of the x term.

(b) State the y-intercepts of each line.

- The y-intercept of the line $y = 3x - 1$ is $b = -1$, the constant term.
- The y-intercept of the line $y = 1 - x$ is $b = 1$, the constant term.

Exercise 9.20. ■

9.21. *Solution*: The point of intersection (x, y) satisfies *both* equations. Generally, to find the point of intersection we equate the y-values:

$$\begin{aligned} 3x - 1 = y = 1 - x &\quad \triangleleft \text{ equate ordinates} \\ 3x - 1 = 1 - x &\quad \triangleleft \text{ eliminate redundant notation} \\ 4x = 2 & \\ x = \frac{1}{2} & \end{aligned}$$

The two lines intersect when the abscissa is $x = \frac{1}{2}$. But

$$x = \frac{1}{2} \implies y = 1 - x\big|_{x=\frac{1}{2}} = 1 - \frac{1}{2} = \frac{1}{2}$$

Thus, the two lines intersect at $\boxed{P(\,\frac{1}{2}, \frac{1}{2}\,)}$

Draw these two lines on the same sheet of paper in order to see the point of intersection. (Or, have your student held graphing calculator do it for you.)

Exercise 9.21. ■

9.22. *Solution*: We simply put each in the *slope-intercept form.*

(a) Find the slope and y-intercept of the line $5x + 3y - 1 = 0$.

$$5x + 3y - 1 = 0$$
$$3y = -5x + 1$$
$$y = -\tfrac{5}{3}x + \tfrac{1}{3} \qquad \triangleleft\ y = mx + b$$

This line has slope $m = -\frac{5}{3}$ and y-intercept of $y = \frac{1}{3}$.

(b) Find the slope and y-intercept of the line $-3x + 12y + 13 = 0$.

$$-3x + 12y + 13 = 0$$
$$12y = 3x - 13$$
$$y = \tfrac{1}{4} - \tfrac{13}{12} \qquad \triangleleft\ y = mx + b$$

This line has slope $m = \frac{1}{4}$ and has y-intercept of $y = -\frac{13}{12}$.

(c) Find the slope and y-intercept of the line $x - y = 0$.

$$x - y = 0$$
$$y = x \quad \triangleleft\ y = mx + b$$

This line has slope $m = 1$ and y-intercept of $y = 0$.

Exercise 9.22. ■

9.23. *Solution*: Trivial. The key point is finding an additional point on the each line—that's done using the equation. Exercise 9.23. ■

9.24. *Solution*: There are two solutions.

The Quick Solution: Note that the two given points, $(-4, 1)$ and $(5, 1)$, have the same *ordinate*; therefore, this is a *horizontal line*. Its equation is

$$\boxed{y = 1}$$

The Long Solution: The long solution uses standard methods.

$$\begin{aligned}
\text{Calculate Slope:} \quad m &= \frac{y_2 - y_1}{x_2 - x_1} \\
m &= \frac{1 - 1}{5 + 4} = 0 \\
\text{Point-Slope Form:} \quad y - y_1 &= m(x - x_1) \\
y - 1 &= 0(x - (-4)) \\
y &= 1
\end{aligned}$$

Again we arrive at the conclusion that $\boxed{y = 1}$ is the equation of the line. Which way did you do it?

Exercise 9.24. ■

9.25. *Solution*: The calculation of the intercepts from the general form is very quick. The graphing is trivial so I'll not bother to prepare the graphs for you.

(a) Find the intercepts of $3x + 4y = 24$.

$$3x + 4y = 24$$

y-intercept: Put $x = 0$	x-intercept: Put $y = 0$
$4y = 24$	$3x = 24$
$y = 6$	$x = 8$

Presentation of Answer: The x- and y- intercepts are, respectively, $\boxed{x = 8, y = 6}$.

The solutions to the other two parts are on the next page. Given that you have looked at the solution to part (a), you may want to revise your solutions to parts (b) and (c) before looking.

(b) Find the intercepts of $5x - 2y = 10$.

$$5x - 2y = 10$$

y-intercept: Put $x = 0$	x-intercept: Put $y = 0$
$-2y = 10$	$5x = 10$
$y = -5$	$x = 2$

Presentation of Answer: The x- and y- intercepts are, respectively, $\boxed{x = 2, y = -5}$.

(c) Find the intercepts of $x - 2y = 1$.

$$x - 2y = 1$$

y-intercept: Put $x = 0$	x-intercept: Put $y = 0$
$-2y = 1$	$x = 1$
$y = -\frac{1}{2}$	

Presentation of Answer: The x- and y- intercepts are, respectively, $\boxed{x = 1, y = -\frac{1}{2}}$.

The graphing of these equations is left to the student—that's you.

Exercise 9.25. ■

9.26. *Solutions*:

(a) Find the point of intersection: $x - y = 1$ and $x + y = 1$.

Solution: Begin by solving for y in each equation.

$x - y = 1$	◁ given	$x + y = 1$	◁ given
$y = x - 1$	◁ solve for y	$y = 1 - x$	◁ solve for y

Now equate ordinates

$$x - 1 = y = 1 - x$$

and solve for x

$$x - 1 = 1 - x \quad \triangleleft \text{ from above}$$
$$2x = 2$$
$$x = 1 \quad \triangleleft \text{ solved!}$$

Now, substitute the value of $x = 1$ into any of the two equations to get $y = 0$.

Presentation of Answer: Point of intersection is $\boxed{P(1, 0)}$

(b) Find the point of intersection: $x - 3y = 1$ and $2x - y = 1$.

$x - 3y = 1$ ◁ given	$2x - y = 1$ ◁ given
$y = \frac{1}{3}(x - 1)$ ◁ solve for y	$y = 2x - 1$ ◁ solve for y

Now equate ordinates

$$\tfrac{1}{3}(x - 1) = y = 2x - 1$$

and solve for x

$$\tfrac{1}{3}(x - 1) = 2x - 1 \quad \triangleleft \text{ from above}$$
$$x - 1 = 6x - 3 \quad \triangleleft \text{ clear fractions}$$
$$x = \tfrac{2}{5} \quad \triangleleft \text{ solve for } x$$

Now, substitute the value of $x = \frac{2}{5}$ into any of the two equations to get $y = -\frac{1}{5}$.

Presentation of Answer: Point of intersection is $\boxed{P(\frac{2}{5}, -\frac{1}{5})}$

(c) Find the point of intersection: $4x - 2y = 7$ and $y = 2x + 1$.

$4x - 2y = 7$	◁ given	$y = 2x + 1$	◁ given
$y = \frac{1}{2}(4x - 7)$	◁ solve for y		

Now equate ordinates

$$\tfrac{1}{2}(4x - 7) = 2x + 1$$

and solve for x

$$\tfrac{1}{2}(4x - 7) = 2x + 1 \quad \triangleleft \text{ from above}$$
$$4x - 7 = 4x + 2 \quad \triangleleft \text{ clear fractions}$$
$$-7 = 2 \quad \triangleleft \text{ substract } 4x \text{ from both sides}$$

Conclusions: The last equation has no solution—the x's have been totally eliminated. This means that there are no x's that satisfy the equation $\frac{1}{2}(4x - 7) = 2x + 1$; hence, these two lines do not intersect—they are in fact parallel.

(d) Find the point of intersection: $6x + y = 3$ and $x + y = 2$.

$6x + y = 3$	◁ given	$x + y = 2$	◁ given
$y = -6x + 3$	◁ solve for y	$y = 2 - x$	◁ solve for y

Now equate ordinates

$$-6x + 31 = y = 2 - x$$

and solve for x

$$\begin{aligned} -6x + 3 &= 2 - x && \triangleleft \text{ from above} \\ -5x &= -1 \\ x &= \tfrac{1}{5} \end{aligned}$$

Now, substitute the value of $x = \frac{1}{5}$ into any of the two equations to get $y = \frac{9}{5}$.

Presentation of Answer: Point of intersection is $\boxed{P(\frac{1}{5}, \frac{9}{5})}$

Exercise 9.26. ■

9.27. *Solution*: This problem is no different from the problems seen earlier (See, e.g., Exercise 9.14); the only difference is that we get our slope from the given line.

The slope of the line $y = 2x - 1$ is $m = 2$. We want to construct a line parallel to this line, so it too should have a slope of $m = 2$; however, our line is required to pass through the point $P(2, 7)$:

$$\text{Given: } m = 2 \text{ and } P(2, 7)$$

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\ y - 7 &= 2(x - 2) && \triangleleft \text{ substitute} \\ y &= 2x + 3 \end{aligned}$$

Presentation of Answer: The line that is parallel to the line $y = 2x - 1$ and passes through the point $P(2, 7)$ is

$$\boxed{y = 2x + 3}$$

Exercise 9.27. ■

9.28. *Solution*: Same problem as just solved, the only difference is we have to work a little harder to get the slope.

$$\begin{aligned} \text{Given line: } 5x + 4y &= 12 \\ 4y &= -5x + 12 \\ y &= -\tfrac{5}{4}x + 12 \end{aligned}$$

The slope of the given line is $m = -\frac{5}{4}$.

$$\text{Given: } m = -\tfrac{5}{4} \text{ and } P(-3, -1)$$

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\ y + 1 &= -\tfrac{5}{4}(x + 3) && \triangleleft \text{ substitute} \\ y &= -\tfrac{5}{4}x - \frac{15}{4} - 1 \\ y &= -\tfrac{5}{4}x - \frac{19}{4} \end{aligned}$$

Presentation of Answer: $\boxed{y = -\tfrac{5}{4}x - \frac{19}{4}}$ or $\boxed{5x + 4y + 19 = 0}$

Exercise 9.28. ■

9.29. When $m_1 = 0$ ($m_2 = 0$), then line ℓ_1 (resp. ℓ_2) is a horizontal line. In this case, only vertical lines are perpendicular to it!

Typically, horizontal lies have equation $y = a$, and vertical lines have equation $x = b$. It is very easy to determine whether any line is perpendicular to either a horizontal line or a vertical line.

Exercise 9.29. ■

9.30. *Solution*: The slope of the given line is $m_{\text{given}} = 5$ and so the slope of the line we are trying to construct has slope

$$m = -\frac{1}{5} \qquad \triangleleft \text{ from (14)}$$

We now go into our point-slope mode (our most powerful mode of thought): Given $m = -\frac{1}{5}$ and $P(3, 4)$,

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\ y - 4 &= -\tfrac{1}{5}(x - 3) && \triangleleft \text{ substitute} \\ y &= 4 - \tfrac{1}{5}x + \tfrac{3}{5} \\ y &= -\tfrac{1}{5}x + \tfrac{23}{5} && \triangleleft \text{ slope-intercept} \end{aligned}$$

Presentation of Solution: The desired line is $\boxed{y = -\frac{1}{5}x + \frac{23}{5}}$

Exercise 9.30. ■

9.31. *Solution*: The given line $6x + 2y = 1$ can be rewritten in the slope-intercept form as $y = -3x + \frac{1}{6}$. Therefore, the given line has slope of $m_{\text{given}} = -3$, so the line we are trying to construct has slope

$$m = -\frac{1}{-3} = \frac{1}{3} \qquad \triangleleft \text{ from (14)}$$

We use the point-slope form: Given $m = \frac{1}{3}$ and $P(-5, 3)$,

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ point-slope form} \\ y - 3 &= \tfrac{1}{3}(x + 5) && \triangleleft \text{ substitute} \\ y &= 3 + \tfrac{1}{3}x + \tfrac{5}{3} \\ y &= \tfrac{1}{3}x + \tfrac{14}{3} && \triangleleft \text{ slope-intercept} \end{aligned}$$

Presentation of Solution: The desired line is $\boxed{y = \frac{1}{3}x + \frac{14}{3}}$

Exercise 9.31. ■

9.32. *Solution to* (a) Let ℓ_1 denote the given line $y = 2x + 4$. Here is an obvious sequence of steps.

1. Compute the equation of the line ℓ_2 that passes through the given point, $P(-1, -1)$ and is perpendicular to the given line, $\ell_1 : y = 2x + 4$. Draw a picture of this line.
2. Find the point of intersection of the two lines ℓ_1, the given line, and ℓ_2, the line perpendicular to the ℓ_1, as developed in *Step* 1.
3. Having found the point of intersection between ℓ_1 and ℓ_2, call this point Q, the desired distance is $d(P, Q)$, the distance between point P and Q.

Carry out this game plan before looking at the solution to part (b) on the next page.

Step 1: Find the equation of the line perpendicular to the given line ℓ_1 and passing through the given point. The equation of that line is

$$y + 1 = -\tfrac{1}{2}(x + 1)$$

or,

$$\boxed{y = -\tfrac{1}{2}x - \tfrac{3}{2}}$$

Verify these calculations if you missed this step.

Step 2: Find the point of intersection between $\ell_1 : y = 2x + 4$ and $\ell_2 : y = -\frac{1}{2}x - \frac{3}{2}$.

$$2x + 4 = y = -\tfrac{1}{2}x - \tfrac{3}{2}$$

therefore,

$$\begin{aligned} \tfrac{5}{2}x &= -\tfrac{11}{2} \\ x &= -\tfrac{11}{5} \end{aligned}$$

Verify all details. Substitute this value of x into either of the two equation to obtain

$$y = 2x + 4\big|_{x=-11/5} = -\tfrac{2}{5}$$

Thus, the point of intersection between ℓ_1 and ℓ_2 is

$$Q\left(-\tfrac{11}{5}, -\tfrac{2}{5}\right)$$

Step 3: Find the distance between P and Q.

Given $P(-1, -1)$ and $Q\left(-\tfrac{11}{5}, -\tfrac{2}{5}\right)$,

$$\begin{aligned} d(P, Q) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-\tfrac{11}{5} + 1)^2 + (-\tfrac{2}{5} + 1)^2} \\ &= \sqrt{(-\tfrac{6}{5})^2 + (\tfrac{3}{5})^2} \\ &= \sqrt{\frac{36}{25} + \frac{9}{25}} \\ &= \tfrac{3}{5}\sqrt{5} \end{aligned}$$

Presentation of Answer: The distance between the point $P(-1,-1)$ and the line $y = 2x + 4$ is

$$\boxed{d(P, \ell_1) = \frac{3}{5}\sqrt{5}}$$

Comments: That was a multistep problem! Each step, however, was a problem type examined earlier. In mathematics, we try to outline in our minds a plan of attack, then try to carry out the plan—being ready and willing to modify our plan or our thinking at all times.

This problem can be carried out in the abstract to obtain a nice little formula.

For those who want to do more. **Problem:** Find a general formula for the problem of finding the distance from a given point $P(x_0, y_0)$ and a given line $y = mx + b$.

Solution: Just carry out the game plan as outlined in part (a).

Exercise 9.32. ■

Solutions to Examples

9.1. *Solution*:

$$
\begin{aligned}
y &= 3x - 1 && \triangleleft \text{ given} \\
3x - 1 &= y && \triangleleft \text{ transpose} \\
3x &= y + 1 \\
x &= \tfrac{1}{3}(y + 1) && \triangleleft \text{ solve for } x
\end{aligned}
$$

Thus, we have written x as a function of y:

$$\boxed{x = \tfrac{1}{3}(y + 1)}$$

Example 9.1. ■

9.2. *Solution*: To calculate slope, you take the difference in the ordinates (the second coordinates) and divide by the difference in the abscissas (the first coordinates) being sure to subtract in the same order in both numerator and denominator.

$$\text{Given: } P(-1,3) \quad Q(4,1)$$

$$m = \frac{3-1}{-1-4} = -\frac{2}{5}$$

The slope of the line is $\boxed{m = -\frac{2}{5}}$ Example 9.2. ■

9.3. *Solution*: We can substitute directly into equation (8). I shall consider $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (6, 9)$.

Thus, from (8) we have

$$\begin{aligned} y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\ y - 4 &= \frac{9 - 4}{6 - (-2)}(x - (-2)) \\ y - 4 &= \frac{5}{8}(x + 2) \end{aligned}$$

An equation for this line is

$$\boxed{y - 4 = \frac{5}{8}(x + 2).}$$

If we write this equation so that y is a function of x we get

$$\boxed{y = 4 + \frac{5}{8}(x + 2).}$$

Example 9.3. ■

9.4. *Solution*: The equation of the line obtained in Example 9.3 is

$$y = 4 + \frac{5}{8}(x + 2). \tag{S-1}$$

Find the y-intercept: The y-intercept is the point on the y-axis at which the line crosses the y-axis. Characteristic of points on the y-axis is that the x-coordinates are equal to zero. Therefore, to answer the question we seek the point on the line for which the x-coordinate is zero.

Set $x = 0$ in (S-1) to obtain the corresponding value of y:

$$y = 4 + \frac{5}{8}(x + 2)\Big|_{x=0} = 4 + \frac{5}{8}(2) = \frac{21}{4}$$

The y-intercept is $\boxed{y = \dfrac{21}{4}}$

Find the x-intercept: The x-intercept is the point where the line crosses the x-axis. Characteristic of points on the x-axis is that $y = 0$. To find the x-intercept we set $y = 0$ in equation (S-1) and solve for x.

$$4 + \frac{5}{8}(x+2) = y \qquad \triangleleft \text{ Now put } y = 0 \text{, and } \ldots$$

$$4 + \frac{5}{8}(x+2) = 0 \qquad \triangleleft \text{ solve for } x$$

$$\frac{5}{8}(x+2) = -4$$

$$x + 2 = -\frac{32}{5}$$

$$x = -\frac{42}{5}$$

Thus the x-intercept is $\boxed{x = -\frac{42}{5}}$

Find the point on the curve corresponding to $x = -2$:

$$
\begin{aligned}
y &= 4 + \frac{5}{8}(x+2)\Big|_{x=-3} \\
&= 4 + \frac{5}{8}(-3+2) = 4 - \frac{5}{8} = \boxed{\frac{27}{8}}
\end{aligned}
$$

Thus, the point on the curve corresponding to $x = -2$ is $\boxed{\left(-2, \tfrac{27}{8}\right)}$

Find the point on the curve corresponding to $x = 6$:

$$
\begin{aligned}
y &= 4 + \frac{5}{8}(x+2)\Big|_{x=6} \\
&= 4 + \frac{5}{8}(6+2) = 4 + 5 = \boxed{9}
\end{aligned}
$$

Thus, the point on the curve corresponding to $x = 6$ is $\boxed{(6, 9)}$

Example 9.4. ■

9.5. *Solution*: We use equation (10):

$$\text{Given: } m = 5 \text{ and } P(1,2)$$

$$\begin{aligned} y - y_1 &= m(x - x_1) && \triangleleft \text{ equation (10)} \\ y - 2 &= 5(x - 1) && \triangleleft \text{ substitute} \\ y &= 2 + 5(x - 1) && \triangleleft \text{ write as a function of } x \\ y &= 5x - 3 && \triangleleft \text{ combine} \end{aligned}$$

Presentation of Solution: $\boxed{y = 5x - 3}$

Comments: As you can see the point-slope was indeed a transitory form. We used it to obtain an initial formula, then manipulated from there.

Example 9.5. ■

9.6. *Solution*: We simply put the equation in the *slope-intercept form.*

$$\begin{aligned} 2x - 3y &= 5 \\ -3y &= -2x + 5 \\ y &= \tfrac{2}{3}x - \tfrac{5}{3} \end{aligned} \tag{S-3}$$

We can now see that the given equation has slope of $\boxed{m = \frac{2}{3}}$ and, by the way, crosses the x-axis at $x = -\frac{5}{3}$ Example 9.6. ■

Important Points

Is x a function of y?

The answer is **Yes**. When we say that x is a function y we mean

$$x = f(y)$$

for some function f; that is, x is expressible in terms of y. This means that corresponding to each y-value (the independent variable), there is a unique x-value (the dependent variable). In FIGURE 5, for any given y, draw a horizontal line at y. Note that this line intersects the curve in at most one location. This means that for any given y there corresponds a single x-value. This is the definition of function.

Important Point ■

Throughout this discussion, we refer to FIGURE 6. The triangle PRQ is similar to the triangle $P'R'Q'$. Therefore the ratio of corresponding sides are equal. Thus,

$$\frac{d(R,Q)}{d(P,R)} = \frac{d(R',Q')}{d(P',R')}$$

or,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2' - y_1'}{x_2' - x_1'}$$

The last set of calculation follow form our discussion of the distances between vertically and horizontally oriented points as presented in LESSON 8.

Note: The above argument is not completely general. Do you see the "weakness" of the argument? Important Point ■

The point $(2, -1)$ is *not* on the line $y = \frac{1}{2}x - 4$ because it *does not satisfy* the equation:

$$\frac{1}{2}x - 4\Big|_{x=2} = -2 \neq -1.$$

Important Point ■