THE UNIVERSITY OF AKRON
Theoretical and Applied Mathematics

Estimating the value of e

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1. Introduction

Let a be a positive number other than 1. The exponential function, base a, is defined as

$$y = a^x, \quad x \in (-\infty, \infty), \quad y \in (0, \infty)$$

The purpose of this student hands-on document is to explore the derivative of the exponential, in particular, estimate the value of the famous natural number e. The details follow in subsequent sections.

2. The Derivative of the Exponential

Let $f(x) = a^x$, where $a > 0$ and different from 1. The derivative of f can be formally computed from the limit of its difference quotient: We have

$$f(x) = a^x \quad f(x+h) = a^{x+h} \tag{1}$$

Now, we build the difference quotient of f:

$$
\begin{aligned}
f(x+h) - f(x) &= a^{x+h} - a^x = a^x a^h - a^x = a^x(a^h - 1) \\
\frac{f(x+h) - f(x)}{h} &= \frac{a^x(a^h - 1)}{h} \qquad (2)
\end{aligned}
$$

Now, the derivative of f is given by

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} && \text{from equation (2)} \\
&= a^x \lim_{h \to 0} \frac{a^h - 1}{h} && \text{factor out } a^x \text{ (limit in } h\text{)} \\
&= C_a a^x && (3)
\end{aligned}
$$

Here, in equation (3), we have defined

$$
C_a = \lim_{h \to 0} \frac{a^h - 1}{h} \qquad (4)
$$

Note that C_a is a constant (not depending on x), but the value of C_a does depend on the value of the base of the exponent, a. The notation, C_a, reflects the dependence on on a.

From equation (3), we have

$$f'(x) = C_a a^x$$

or

$$\frac{d}{dx} a^x = C_a a^x \tag{5}$$

where C_a is an unknown constant the value of which depends on a.

3. Estimating the value of C_a

In this section, we give a hands-on demo of estimating the values for C_a, for various values of the base a. The idea is very simple, we use equation (4): when h is close to zero, we should have

$$C_a \approx \frac{a^h - 1}{h}$$

The demo begins on the next page: ▶

Instructions: Enter a base value, a, $a > 0$ and $a \neq 1$, then press the `Enter` key to "commit" the value.

$a =$

Table of values of $\dfrac{a^h - 1}{h}$

Estimate: $a =$

$C_a \approx$

The demo continues on the next page: ▶

4. Estimating e

One definition of the natural number e is that it is that value where $C_e = 1$. See equation (4) for the definition of C_a, for any base number a. In this case, from equation (5), the derivative of the exponential with a base of e is given by

$$\frac{d}{dx} e^x = e^x$$

The interactive demo of the previous page can be used to "estimate" e; however, on this page, we present an simplified version.

Instructions: Enter a base value, a, $a > 0$ and $a \neq 1$, then press the `Enter` key to "commit" the value. Try to get the estimated value of C_a as close to 1 as you have the patience to try.

$a =$ [ ]

$C_a =$

This value of a is ...

Give Up?