THE UNIVERSITY OF AKRON
Mathematics and Computer Science

⇑

calculus 1
spring '97

Article: Friday Quizzes

This file contains the Friday, in-class quizzes in *Calculus I*.

Directory

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Friday Quizzes

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Quiz/Solutions: Week 1.

No quiz was given for this week.

Quiz/Solutions: Week 2.

Quiz #1

1. Define $f(x) = \sqrt{x^2 - 2}$. Calculate the domain of f.

Solution: The reasoning would be as follows: for the function

$$f(x) = \sqrt{x^2 - 2},$$

we require

$$x^2 - 2 \geq 0.$$

Now for the task of solving such an inequality. We use standard techniques.

$$\begin{aligned} x^2 - 2 \geq 0 &\iff x^2 \geq 2 \\ &\iff |x| \geq \sqrt{2} \end{aligned}$$

Once the inequality has been solved, the correct interpretation and specification of the domain is necessary.

$$\begin{aligned} \operatorname{Dom}(f) &= \{\, x : |x| \ge \sqrt{2}\,\} \\ &= \{\, x : x \le -\sqrt{2} \text{ or } x \ge \sqrt{2}\} \\ &= (\,-\infty, -\sqrt{2}\,] \cup [\,\sqrt{2}, +\infty\,) \end{aligned}$$

Presentation of Answer:

$$\boxed{\operatorname{Dom}(f) = (\,-\infty, -\sqrt{2}\,] \cup [\,\sqrt{2}, +\infty\,)} \tag{1}$$

Solution Notes: Of the 34 students who took this quiz, 23 missed this problem. Virtually all of the 23 missed it because of algebra; in particular, this problem was missed because people didn't know that

$$x^2 \ge 2 \text{ is equivalent to } |x| \ge \sqrt{2}$$

and this, in turn, is equivalent to

$$\underbrace{x \leq -\sqrt{2}}_{\substack{\text{all 23 missed} \\ \text{this one}}} \text{ or } x \geq \sqrt{2}.$$

These specifications lead then to the answer given in (1).

▪ Of the remaining 11 students who did not make such a fundamental error, four of them made this error

$$x^2 \geq 2 \implies x \geq \pm 2.$$

This is an algebraically incorrect statement, but by whatever reasoning processes the students employed, they "got" the correct answer, as given in (1).

▪ Really, only four students solved this problem correctly using good solid algebraic techniques.

▪ Needless to say, everyone should work on their algebra. ▪

2. Calculate $f(\sqrt{x})$.

Solution: Given that $f(x) = \sqrt{x^2 - 2}$, we just use the replacement method.

$$f(\sqrt{x}) = \sqrt{(\sqrt{x})^2 - 2} = \sqrt{x - 2}$$

Presentation of Answer:

$$\boxed{f(x) = \sqrt{x - 2}}$$

Solution Notes: Most everyone received full credit for this problem with one or two exceptions. ■

3. Define a function g by $g(x) = f(\sqrt{x})$. Calculate the domain of g.

Solution: From problem #1, $g(x) = f(\sqrt{x}) = \sqrt{x - 2}$. The calculation of this function requires $x - 2 \geq 0$; therefore $x \geq 2$.

Presentation of Answer:

$$\boxed{\operatorname{Dom}(g) = [\,2, +\infty\,)}$$

Quiz/Solutions: Week 3.

Quiz #2

1. Calculate $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}}{3x-2}$.

Solution: Notice that the limit of the denominator is nonzero; therefore, this problem is quite simple:

$$\lim_{x\to 2} \frac{\sqrt{x^2+5}}{3x-2} = \frac{\sqrt{2^2+5}}{3(2)-2} = \frac{3}{4}$$

Presentation of Answer:

$$\boxed{\lim_{x\to 2} \frac{\sqrt{x^2+5}}{3x-2} = \frac{3}{4}.}$$

2. Calculate $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x - 1}$.

Solution: The first thing to observe is that the limit of the denominator is zero. The next thing to check is the limit of the numerator—also zero. Therefore, the strategy, at this level of play, is to seek a common factor to cancel.

Algebraic Sidebar: Notice that, for $x \neq 1$, we have

$$\frac{x^2 - 3x + 2}{x - 1} = \frac{(x - 1)(x - 2)}{x - 1} = x - 2 \qquad (1)$$

The Main Problem:

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1} &= \lim_{x \to 1} (x - 2) \qquad \triangleleft \text{ from } (1) \\ &= 1 - 2 = -1 \end{aligned}$$

Presentation of Answer:

$$\boxed{\lim_{x\to 1} \frac{x^2 - 3x + 2}{x - 1} = -1.}$$

3. Define a function

$$f(x) = \begin{cases} 2x + 1 & x < 1 \\ x^3 - 2 & x \geq 1 \end{cases}$$

Discuss $\lim_{x\to 1} f(x)$.

Solution: This is a piecewise defined function, and the limit of interest is at the point at which we are 'piecing' two functions together, i.e., at $x = 1$. Therefore, a *one-sided approach* is called for.

The Left-Hand Limit:

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (2x + 1) && \triangleleft \text{ since } x < 1 \text{ here} \\ &= 3. \end{aligned}$$

Thus,

$$\lim_{x \to 1^-} f(x) = 3. \tag{2}$$

In a similar manner, we calculate the ...

Right-Hand Limit:

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^-} (x^3 - 2) && \triangleleft \text{ since } x > 1 \text{ here} \\ &= -1. \end{aligned}$$

Thus,

$$\lim_{x \to 1^+} f(x) = -1. \tag{3}$$

Discussion of limit: We summarize out calculations,

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= 3 && \triangleleft \text{ from } (2) \\ \lim_{x \to 1^+} f(x) &= -1 && \triangleleft \text{ from } (3) \end{aligned}$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we conclude that the two-sided limit does not exist, i.e.,

$$\boxed{\lim_{x\to 1} f(x) \text{ d.n.e.}}$$

Quiz/Solutions: Week 4.

Quiz #3

1. Calculate $\lim_{x\to 2} \dfrac{\sqrt{9x-2}-4}{x-2}$.

Solution: Notice that both the numerator and denominator tend to zero. In this situation, we need to remove some "hidden" factors.

Alegbraic Preliminaries:

$$\begin{aligned}
\frac{\sqrt{9x-2}-4}{x-2} &= \frac{\sqrt{9x-2}-4}{x-2}\,\frac{\sqrt{9x-2}+4}{\sqrt{9x-2}+4} && \triangleleft \text{ Multiply by conjugate} \\
&= \frac{(9x-2)-16}{(x-2)\sqrt{9x-2}+4} \\
&= \frac{9x-18}{(x-2)\sqrt{9x-2}+4} \\
&= \frac{9(x-2)}{(x-2)\sqrt{9x-2}+4} && \triangleleft \text{ Cancel!}
\end{aligned}$$

Thus,

$$\frac{\sqrt{9x-2}-4}{x-2} = \frac{9}{\sqrt{9x-2}+4} \tag{1}$$

Now we are ready to calculate the limit.

$$\begin{aligned} \lim_{x\to 2} \frac{\sqrt{9x-2}-4}{x-2} &= \lim_{x\to 2} \frac{9}{\sqrt{9x-2}+4} && \triangleleft \text{ by (1)} \\ &= \frac{9}{8}. && \triangleleft \text{ Skill Level 0} \end{aligned}$$

Presentation of Answer: Therefore,

$$\boxed{\lim_{x\to 2} \frac{\sqrt{9x-2}-4}{x-2} = \frac{9}{8}.}$$

2. Prove: $\lim_{x \to 2} 5x + 1 = 11$.

Solution: Let $\epsilon > 0$. Choose $\delta = \epsilon/5$. Now, let x be any number such that $0 < |x - 2| < \delta$, then it follows that

$$|(5x + 1) - 11| = |5x - 10| = 5|x - 2| < 5\delta = 5\frac{\epsilon}{5} = \epsilon.$$

Thus we have shown that for any number $\epsilon > 0$, there exists a number $\delta > 0$ ($\delta = \epsilon/5$), such that

$$0 < |x - 2| < \delta \implies |(5x + 1) - 11| < \epsilon.$$

This is precisely what we needed to prove that $\lim_{x \to 2} 5x + 1 = 11$. $\square$

Quiz/Solutions: Week 5.

No quiz given due the existence of Test #1.

Quiz/Solutions: Week 6.

Quiz #4

1. Use the definition of derivative to calculate the derivative of the function $f(x) = 2/x$.

Solution: First calculate the simplify the *difference quotient*:

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}[f(x+h)-f(x)] && \triangleleft \text{ this makes it easier} \\
&= \frac{1}{h}\left[\frac{2}{x+h}-\frac{2}{x}\right] && \triangleleft \text{ replacement technique} \\
&= \frac{2}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] && \triangleleft \text{ get common denom.} \\
&= \frac{2}{h}\left[\frac{-h}{x(x+h)}\right] && \triangleleft \text{ combine} \\
&= \frac{-2h}{hx(x+h)}
\end{aligned}
$$

Cancelling the factor of h is last expression, we obtain,

$$\frac{f(x+h)-f(x)}{h} = \frac{-2}{x(x+h)} \tag{1}$$

Finally,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{-2}{x(x+h)} && \triangleleft \text{ from (1)} \\ &= -\frac{2}{x^2} \end{aligned}$$

Presentation of Answer: $\boxed{f'(x) = -\frac{2}{x^2}.}$

2. Define a function

$$f(x) = \begin{cases} 2x+1 & x < 1 \\ x^3 - 2 & x \geq 1 \end{cases}$$

Discuss $\lim_{x \to 1} f(x)$.

Solution: This is a piecewise defined function, and the limit of interest is at the point at which we are 'piecing' two functions together, i.e., at $x = 1$. Therefore, a *one-sided approach* is called for.

The Left-Hand Limit:

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (2x+1) \qquad \triangleleft \text{ since } x < 1 \text{ here} \\ &= 3. \end{aligned}$$

Thus,

$$\lim_{x \to 1^-} f(x) = 3. \tag{2}$$

In a similar manner, we calculate the ...

Right-Hand Limit:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} (x^3 - 2) \qquad \triangleleft \text{ since } x > 1 \text{ here}$$
$$= -1.$$

Thus,

$$\lim_{x\to 1^+} f(x) = -1. \tag{3}$$

Discussion of limit: We summarize out calculations,

$$\lim_{x\to 1^-} f(x) = 3 \qquad \triangleleft \text{ from (2)}$$
$$\lim_{x\to 1^+} f(x) = -1 \qquad \triangleleft \text{ from (3)}$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we conclude that the two-sided limit does not exist, i.e.,

$$\boxed{\lim_{x\to 1} f(x) \text{ d.n.e.}}$$

Quiz/Solutions: Week 7.

Quiz #5

1. Calculate $\dfrac{d}{dx}\tan(3x^2)$.

Solution: Initially, you should analyze this as the derivative of the tangent of some function of x. Thus,

$$\begin{aligned} \frac{d}{dx}\tan(3x^2) &= \sec^2(3x^2)\frac{d}{dx}3x^2 && \triangleleft \text{ from Trig. Diff. Formulas.} \\ &= \sec^2(3x^2)(6x) && \triangleleft \text{ by the Power Rule} \end{aligned}$$

Now rearranging the above calculation in a more esthetically pleasing form, we get

$$\boxed{\frac{d}{dx}\tan(3x^2) = 6x\sec^2(3x^2)}$$

Problem Notes: This is a Skill Level 1 problem. It's a simple exmaple of recognition and implementation. ∎

2. Calculate $\dfrac{d}{dx}[\sin(\cos(x))]^3$.

Solution: The function to be differentiated is more commonly written as

$$[\sin(\cos(x))]^3 = \sin^3(\cos(x))$$

This enables us to write the function with one less set of brackets. We are asked initially to differentiate a *power function*. We'll use the Power Rule.

$$\begin{aligned}
&\frac{d}{dx}[\sin(\cos(x))]^3 \\
&\quad = 3[\sin(\cos(x))]^2 \frac{d}{dx}\sin(\cos(x)) && \triangleleft \text{ by the power rule} \\
&\quad = 3[\sin(\cos(x))]^2 \cos(\cos(x)) \frac{d}{dx}\cos(x) && \triangleleft \text{ Trig. Diff. Formulas} \\
&\quad = 3[\sin(\cos(x))]^2 (\cos(\cos(x))(-\sin(x)) && \triangleleft \text{ Trig. Diff. Formulas} \\
&\quad = -3\sin(x)\cos(\cos(x))[\sin(\cos(x))]^2
\end{aligned}$$

Presentation of Answer:

$$\boxed{\frac{d}{dx}[\sin(\cos(x))]^3 = -3\sin(x)\cos(\cos(x))\sin^2(\cos(x)).}$$

3. Calculate $\dfrac{d}{dx}[(2x+1)^3(4x+5)^2]$,

Solution: This is a simple product rule/power rule problem. We begin with the Product Rule,

$$\begin{aligned}
&\frac{d}{dx}[(2x+1)^3(4x+5)^2] \\
&\qquad = (2x+1)^3\frac{d}{dx}(4x+5)^2 + (4x+5)^2\frac{d}{dx}(2x+1)^3 \\
&\qquad = (2x+1)^3[2(4x+5)(4)] + (4x+5)^2[3(2x+1)^2(2)]
\end{aligned}$$

This finishes the *Calculus step*. But the above “answer” is offensive to the eye. I want to be true to my algebraic roots and simplify.

$$\begin{aligned}
\frac{d}{dx}&[(2x+1)^3(4x+5)^2] \\
&= (2x+1)^3[2(4x+5)(4)] + (4x+5)^2[3(2x+1)^2(2)] \\
&= 8(2x+1)^3(4x+5) + 6(4x+5)^2(2x+1)^2 \\
&= 2(2x+1)^2(4x+5)[4(2x+1) + 3(4x+5)] \\
&= 2(2x+1)^2(4x+5)(20x+19) \\
&= 2(4x+5)(20x+19)(2x+1)^2
\end{aligned}$$

Presentation of Answer:

$$\frac{d}{dx}[(2x+1)^3(4x+5)^2] = 2(4x+5)(20x+19)(2x+1)^2.$$

Problem Notes: Notice that I have written the lowest order factors first—its an algebraic convention.

- Many people did not know how to do such an elementary simplification; they will have to learn. ■

Quiz/Solutions: Week 8.

Quiz #6 **Note: Test #2 Friday over Chapter #2**

1. Let $xy^2 + \sin(y) = 1$. Calculate $\dfrac{dy}{dx}$.

Solution: We follow the steps of implicit differentiation:

$$
\begin{aligned}
xy^2 + \sin(y) &= 1 && \triangleleft \text{ treat } y \text{ as a function of } x \\
\frac{d}{dx}(xy^2 + \sin(y)) &= 0 && \triangleleft \text{ differentiate both sides} \\
2xyy' + y^2 + \cos(y)y' &= 0 && \triangleleft \text{ apply standard diff. formulas} \\
(2xy + \cos(y))y' &= -y^2 && \triangleleft \text{ re-group}
\end{aligned}
$$

We now solve for y' in the last equation:

$$\boxed{y' = -\frac{y^2}{2xy + \cos(y)}}$$

2. Let $y = \sqrt{3x+1}$. Calculate $\dfrac{d^3y}{dx^3}$.

Solution: We carefully apply the power rule:

$$\begin{aligned} y &= (3x+1)^{1/2} \\ y' &= \frac{1}{2}(3x+1)^{-1/2}\,(3) = \frac{3}{2}(3x+1)^{-1/2} \\ y'' &= -\frac{3}{4}(3x+1)^{-3/2}\,(3) = -\frac{9}{4}(3x+1)^{-3/2} \\ y''' &= \frac{27}{8}(3x+1)^{-5/2}\,(3) = \frac{81}{8}(3x+1)^{-5/2} \end{aligned}$$

Presentation of Answer:

$$\boxed{\frac{d^3y}{dx^3} = \frac{81}{8}(3x+1)^{-5/2}}$$

3. Let $x^3 + y^3 = 1$. Use implicit differentiation to calculate $\dfrac{d^2y}{dx^2}$.

Solution: We apply implicit differentiation *twice*!

Calculation of the First Derivative:

$$x^3 + y^3 = 1 \qquad \triangleleft \text{ given equation} \tag{1}$$

$$\frac{d}{dx}(x^3 + y^3) = 0 \qquad \triangleleft \text{ differentiate both sides}$$

$$3x^2 + 3y^2y' = 0 \qquad \triangleleft \text{ apply diff. rules}$$

$$y' = -\frac{x^2}{y^2} \qquad \triangleleft \text{ solve for } y' \tag{2}$$

Now let's calculate the second dervative: $y'' = \dfrac{dy'}{dx}$,

$$
\begin{aligned}
y'' = \frac{dy'}{dx} &= -\frac{d}{dx}\frac{x^2}{y^2} && \triangleleft \text{ substitute using (2)} \\
&= -\frac{y^2(2x) - x^2(2yy')}{y^4} && \triangleleft \text{ quotient rule} \\
&= -\frac{2xy(y - xy')}{y^4} && \triangleleft \text{ factor} \\
&= -\frac{2x(y - x\frac{-x^2}{y^2})}{y^3} && \triangleleft \text{ substitute using (2)} \\
&= -\frac{2x(y^3 + x^3)}{y^5} && \triangleleft \text{ simplify}
\end{aligned}
$$

But $x^3 + y^3 = 1$ from (1), thus,

$$
\boxed{\frac{d^2y}{dx^2} = y'' = -\frac{2x}{y^5}}
$$

Quiz/Solutions: Week 9.

No quiz due to the occurrence of **Test #2**.

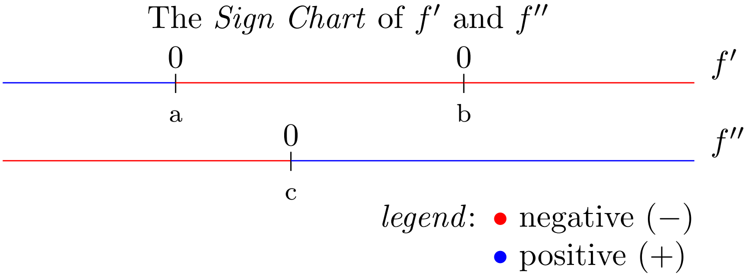
Quiz/Solutions: Week 10.

No quiz was given. Homework Assignment #2 was due instead.

Quiz/Solutions: Week 11.

Quiz #7

1. Consider the following *sign charts*.



Find the intervals of increase and decrease, find where f is concave up and where f is concave down. Finally, identify the numbers a, b, and c as relative maximums, relative minimums, or points of inflection.

Solution: We begin the analysis of this problem with the first derivative.

Monotonicity:

- f is increasing on $(-\infty, a)$ since $f'(x) > 0$ on this interval.
- f is decreasing on (a, b) and on (b, ∞) since $f'(x) < 0$ on each of these intervals.

Concavity: Now for the second derivative information.

- f is concave down on $(-\infty, c)$ since $f''(x) < 0$ over this interval.
- f is concave up on (c, ∞) since $f''(x) > 0$ here.

Classification of a, b, and c.

- f has a relative *maximum* at $x = a$ since $f' > 0$ to the left of a and $f' < 0$ to the right of a.
- The point b is a *saddle point* of f since f' does not change signs at this point. You might term this a *decrasing saddle point*, or maybe not.
- The point c is a *point of inflection* of f since the second derivative changes signs there.

□

2. Consider the function $f(x) = x^3 - 3x$ defined over the interval $[-2, 3]$. Find and classify the *absolute extrema*.

Solution: We find the critical points of f,

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

It is clear from this calculation that f has two critical points:

$$\text{Critical Points: } x = \pm 1.$$

Note that both points lie in the given interval. Now we toss the end-points into the mix:

$$\text{Test Points: } -2, -1, 1, 3$$

and calculate f at each of these points.

Now let's put all the information into a table for greater organization.

x	$f(x)$	
-2	-2	$\Longleftarrow \min f(x)$
-1	2	
1	-2	$\Longleftarrow \min f(x)$
3	18	$\Longleftarrow \max f(x)$

Quiz/Solutions: Week 12.

Test #3 given, so no quiz was given for this week.

Quiz/Solutions: Week 13.

Howework Assignment #3 due, no quiz this week.

Quiz/Solutions: Week 14.

Quiz #8

1. Evaluate $\int_{-1}^{2} 3x^2 - 4x\,dx$.

Solution:

$$\begin{aligned}\int_{-1}^{2} 3x^2 - 4x\,dx &= x^3 - 2x^2\Big|_{-1}^{2} \\ &= \left(2^3 - 2(2^2)\right) - \left((-1)^3 - 2(-1)^2\right) \\ &= (8-8) - (-1-2) \\ &= 3\end{aligned}$$

The Answer: $\boxed{\int_{-1}^{2} 3x^2 - 4x\,dx = 3}$

2. Evaluate $\displaystyle\int \frac{x^3+\sqrt{x}}{x^2\sqrt{x}}\,dx$.

Solution:

$$\begin{aligned}\int \frac{x^3+\sqrt{x}}{x^2\sqrt{x}}\,dx &= \int \frac{x^3}{x^2\sqrt{x}} + \frac{\sqrt{x}}{x^2\sqrt{x}} \\ &= \int x^{1/2} + x^{-2}\,dx \\ &= \frac{2}{3}x^{3/2} - x^{-1} + C\end{aligned}$$

The Answer: $\boxed{\displaystyle\int \frac{x^3+\sqrt{x}}{x^2\sqrt{x}}\,dx = \frac{2}{3}x^{3/2} - \frac{1}{x} + C}$

3. Evaluate $\displaystyle\int (4x^2+1)^{3/2}\, x\, dx$.

Solution: This problem can be solved by the Power Rule.

$$\begin{aligned}
\int (4x^2+1)^{3/2}\, x\, dx &= \frac{1}{8}\int (4x^2+1)^{3/2}\, 8x\, dx && \triangleleft \text{ Let } u = 4x^2+1,\ du = 8x\, dx \\
&= \frac{1}{8}\int u^{3/2}\, du && \triangleleft \text{ Make Substitution} \\
&= \frac{1}{8}\,\frac{2}{5}u^{5/2} + C && \triangleleft \text{ Power Rule} \\
&= \frac{1}{20}(4x^2+1)^{5/2} + C && \triangleleft \text{ Resubstitute}
\end{aligned}$$

The Answer: $$\boxed{\int (4x^2+1)^{3/2}\, x\, dx = \frac{1}{20}(4x^2+1)^{5/2} + C}$$

Quiz/Solutions: Week 15.

No quiz was given for this week.

Important Points

Normally, the reader of the proof does not see side-calculations, there are several reasons for this: (1) they have no direct bearing on the natural flow of the proof; (2) it makes the proof less cluttered; and (3) it leaves the reader wondering: "Where did he/she get that δ value? Gee, I would have never thought of that!"

The last reason is the best!

Side Calculation: Ultimately, we want

$$|f(x) - L| < \epsilon$$

or,

$$|(5x + 1) - 11| < \epsilon. \qquad \text{(I-1)}$$

We now investigate under what condition on x would it possible for (I-1) to be true; in other words, we want to *solve* the inequality (I-1) for x.

Indeed,

$$\begin{aligned} |(5x+1)-11|<\epsilon &\iff |5x-10|<\epsilon \\ &\iff 2|x-2|<\epsilon \\ &\iff |x-2|<\frac{\epsilon}{5} \end{aligned}$$

Thus,

$$|(5x+1)-11|<\epsilon \iff |x-2|<\frac{\epsilon}{5} \tag{I-2}$$

The last line, (I-2), answers the question for us. The inequality

$$|x-2|<\frac{\epsilon}{5}$$

means that x must be less than $\epsilon/5$ away from 2 in order for

$$|(5x+1)-11|<\epsilon$$

to be true. This tells us what we want to choose as the value of δ.

Therefore we say, "Choose $\delta = \dfrac{\epsilon}{5}$." Important Point ■