THE UNIVERSITY OF AKRON
Mathematics and Computer Science

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calculus 1
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Article: Friday Quizzes

This file contains the Friday, in-class quizzes in *Calculus I*.

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- Quizzes and Solutions

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Friday Quizzes

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Quiz/Solutions: Week 1.

Due to the shortened school week, no quiz was given for this week.

Quiz/Solutions: Week 2.

Quiz #1

1. Define $f(x) = \dfrac{x}{1-x^2}$. Calculate each of the following.
   a. $f(4)$;
   b. $f(t^2+1)$;
   c. $f(\sin(x))$;

Solution: The calculations are as follows:

$$f(4) = \frac{4}{1-16} = \boxed{-\frac{4}{15}}$$

$$\begin{aligned} f(t^2+1) &= \frac{t^2+1}{1-(t^2+1)^2} \\ &= \frac{t^2+1}{1-(t^4+2t^2+1)} \\ &= \frac{t^2+1}{-t^4-2t^2} \end{aligned}$$

$$= \boxed{-\frac{t^2+1}{t^4+2t^2}}$$

$$\begin{aligned} f(\sin(x)) &= \frac{\sin(x)}{1-\sin^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \boxed{\sec(x)\tan(x)} \end{aligned}$$

There was a disturbing large number of algebraic errors throughout these little exercises. If you were one of the people who had written on your quiz paper "Bad Algebra," pay particular attention to these solutions.

2. Define the function $F(x) = \dfrac{x^2}{x^2+4}$. Express $F(x)$ in the form $f(g(x))$, for some functions f and g.

Solution: We must think of the composition process and how to produce the function F. After many minutes of meditation we arrive at

$$f(x) = \frac{x}{x+4}$$
$$g(x) = x^2$$

We can check our choices by calculating

$$f(g(x)) = f(x^2) = \frac{x^2}{x^2+4}$$

by replacing x with x^2 in the definition of f.

The key point to observe in the function $F(x)$ was that the only time x appeared was in the form of x^2; this would suggest a two-step calculation process: Take an x, calculate x^2, then take this quantity and perform the calculation $a/(a+4)$ replacing a by x^2.

3. Define the function $f(x) = \dfrac{1}{\sqrt{(x-1)(x+2)}}$. Use the *Sign Chart Method* to find the domain of this function.

Solution: Many people did not use the sign chart method — some of these errored, and some did not. Those that did use a sign chart method — some presented it clearly, others did not (and many times, they errored).

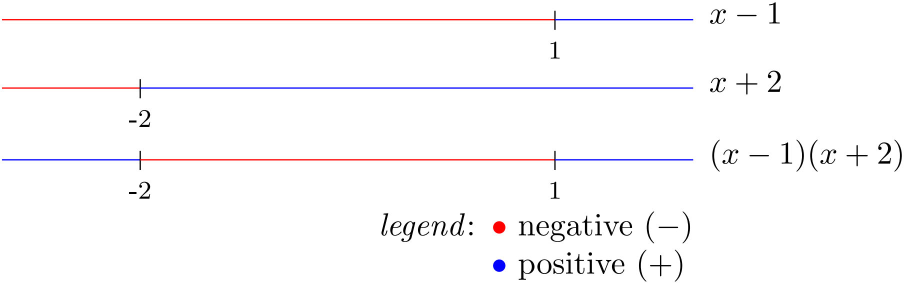
The natural domain of the function $f(x) = \dfrac{1}{\sqrt{(x-1)(x+2)}}$ is all numbers x for which $(x-1)(x+2) > 0$. We exclude the possibility that $(x-1)(x+2) > 0$ be equal to zero as this would make the denominator zero — a no-no!

In order to solve the inequality

$$(x-1)(x+2) > 0$$

we use the *sign-chart method*.

The *Sign Chart* of $(x-1)(x+2)$



It is now clear from the sign chart, that

$$\boxed{\text{Dom}(f) = (\,-\infty, -2\,) \cup (\,1, \infty\,)}$$

Quiz/Solutions: Week 3.

Quiz #2

1. Calculate $\lim_{x\to -1} \dfrac{\sqrt{x^2-4x+4}}{x-1}$.

Solution: The first thing you look at when dealing with a quotient of two functions is the limit of the denominator:

$$\lim_{x\to -1}(x-1) = -1-1 = -2 \neq 0.$$

Therefore, this is a simple problem. The solution is

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2-4x+4}}{x-1} &= \frac{\lim_{x\to -1}\sqrt{x^2-4x+4}}{\lim_{x\to -1}(x-1)} && \triangleleft \text{ Lim. Quot.} \\
&= \frac{\sqrt{\lim_{x\to -1}(x^2-4x+4)}}{-2} && \triangleleft \text{ Cont. Rad.} \\
&= \frac{\sqrt{1+4+4}}{-2}
\end{aligned}$$

$$= \frac{\sqrt{9}}{-2} = -\frac{3}{2}$$

Thus,

$$\boxed{\lim_{x \to -1} \frac{\sqrt{x^2 - 4x + 4}}{x - 1} = -\frac{3}{2}.}$$

Solution Notes: Here are some additional comments and an alternate solution.

Important. Several people made a fundamental algebraic error causing them not to calculate the limit correctly. The following is wrong!

$$\sqrt{x^2 - 4x + 4} = \sqrt{(x-2)^2} \underbrace{=}_{error!} x - 2.$$

The result of this error is to calculate the limit to be $3/2$ instead of $-3/2$. Here is the fundamental fact that needs to be kept ever in mind:

$$\boxed{\sqrt{N^2} = |N|, \text{ for any number } N;}$$

thus,

$$\sqrt{(x-2)^2} = |x-2|$$

would be correct way of taking advantage of the perfect square.

In this case, this alternate approach leads to ...

$$\begin{aligned}\lim_{x\to -1} \frac{|x-2|}{x-1} &= \frac{\lim\limits_{x\to -1} |x-2|}{\lim\limits_{x\to -1} (x-1)} \quad \triangleleft \text{ Lim. Quot.} \\ &= \frac{|-1-2|}{-2} \\ &= \frac{3}{-2} \\ &= \boxed{-\frac{3}{2}}\end{aligned}$$

... the same "answer."

2. Calculate $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution: In this problem, the limit of the denominator is clear zero. Inspecting the numerator, we see that it too tends to zero. We now seek a common factor for the purpose of cancelling out the offending factors.

Algebraic Sidebar:

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2, \qquad x \neq 2$$

Calculation:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

Thus,

$$\boxed{\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.}$$

3. Calculate $\lim_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$.

Solution: In this problem, the denominator tends to zero and the numerator …

$$\lim_{x\to 0} \sqrt{4+x} = \sqrt{\lim_{x\to 0}(4+x)} = \sqrt{4} = 2 \neq 0$$

tends to zero, too. Therefore, we seek a hidden factor to cancel out.

Aux. Calc.: In the presence of the radicals, one would “naturally” think of the *conjugate trick*:

$$\begin{aligned}
\frac{\sqrt{4+x}-2}{x} &= \frac{\sqrt{4+x}-2}{x}\,\frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \\
&= \frac{(4+x)-4}{x(\sqrt{4+x}+2)} \\
&= \frac{x}{x(\sqrt{4+x}+2)} \\
&= \frac{1}{\sqrt{4+x}+2} \qquad \triangleleft \text{ Cancel!}
\end{aligned}$$

Calculation:

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0}\frac{1}{\sqrt{4+x}+2} \qquad \triangleleft \text{ Aux. Calc.}\\ &= \frac{1}{\sqrt{4}+2}\\ &= \frac{1}{4}.\end{aligned}$$

Thus,

$$\boxed{\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} = \frac{1}{4}.}$$

Quiz/Solutions: Week 4.

Quiz #3

1. Find all x at which the function

$$f(x) = \frac{\sqrt{x}}{\sqrt{4-x}}$$

   is *continuous*.

Solution: The set of all x at which this *algebraic function* is continuous is simply its *natural domain*.

In order to take roots, we require $x \geq 0$ and $4 - x \geq 0$. But to avoid dividing by zero, we also require $x \neq 4$. Correlating all this information we arrive at the statement that

$$\boxed{\mathrm{Dom}(f) = [\,0, 4\,)}$$

2. Use the ϵ, δ definition of limit to prove that

$$\lim_{x \to 1} (3x + 2) = 5.$$

Solution: Let $\epsilon > 0$. Choose $\delta = \epsilon/3$, then

$$\begin{aligned} 0 < |x-1| < \delta &\implies |x-1| < \frac{\epsilon}{3} \\ &\implies 3|x-1| < \epsilon \\ &\implies |3x-3| < \epsilon \\ &\implies |(3x+2)-5| < \epsilon \end{aligned}$$

This finishes the proof. $\square$

Solutions Notes: The value of $\delta = \epsilon/3$ was arrived at in the following manner:

$$|(3x+2)-5| = |3x-3| = 3|x-1| < 3\delta$$

We require $3\delta = \epsilon$, or $\delta = \epsilon/3$. Note that I have chosen a different thought pattern than the one usually presented in class.

Quiz/Solutions: Week 5.

No quiz do to the occurrence of Test #1

Quiz/Solutions: Week 6.

Quiz #4

1. Calculate $\dfrac{d}{dx}(\frac{4}{3}x^6 - 5x^2 + 6x^{-8})$.

Solution: This is a simple application of the Power Rule for differentiation.

$$\begin{aligned}\frac{d}{dx}\frac{4}{3}x^6 - 5x^2 + 6x^{-8} &= \frac{4}{3}(6x^5) - 5(2x) + 6(-8x^{-9}) \\ &= \boxed{8x^5 - 10x - 48x^{-9}.}\end{aligned}$$

2. Calculate $\dfrac{d}{dx}(\dfrac{x + 2x^2}{\sqrt{x}})$.

Solution:

$$\frac{d}{dx}\left(\frac{x+2x^2}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2}+2x^{3/2})$$
$$= \boxed{\frac{1}{2}x^{-1/2}+3x^{1/2}.}$$

Now if you are an algebra fanatic, would have continued to simplify.

$$\boxed{\frac{d}{dx}\left(\frac{x+2x^2}{\sqrt{x}}\right) = \frac{1+6x}{2\sqrt{x}}.}$$

3. Calculate $\displaystyle\frac{d}{dx}\frac{x^5}{x^3-2}$.

Solution: This is a job for the quotient rule.

$$\frac{d}{dx}\frac{x^5}{x^3-2} = \frac{(x^3-2)\frac{d}{dx}x^5 - x^5\frac{d}{dx}(x^3-2)}{(x^3-2)^2}$$

$$
\begin{aligned}
&= \frac{(x^3-2)(5x^4) - x^5(3x^2)}{(x^3-2)^2} \\
&= \frac{5x^7 - 10x^4 - 3x^7}{(x^3-2)^2} \\
&= \frac{2x^7 - 10x^4}{(x^3-2)^2} = \boxed{\frac{2x^4(x^3-5)}{(x^3-2)^2}.}
\end{aligned}
$$

Thus,

$$
\boxed{\frac{d}{dx}\frac{x^5}{x^3-2} = \frac{2x^4(x^3-5)}{(x^3-2)^2}.}
$$

Quiz/Solutions: Week 7.

Quiz #5

1. Calculate $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(x/3)}$.

Solution: Naturally, you would use the limit result:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

We must divide the numerator and denominator by x, then multiply in the appropriate "fudge factors."

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(2x)}{\sin(x/3)} &= \lim_{x\to 0} \frac{\sin(2x)/x}{\sin(x/3)/x} \\
&= \lim_{x\to 0} \frac{2\sin(2x)/(2x)}{(1/3)\sin(x/2)/(x/3)} \\
&= \frac{2}{1/3} \lim_{x\to 0} \frac{\sin(2x)/(2x)}{\sin(x/2)/(x/3)} \\
&= \boxed{6.}
\end{aligned}$$

2. Calculate $\dfrac{d}{dx}(4x^4 - 1)^3/7$.

Solution: We have the derivative of a function raised to a power. We'll use the Power Rule:

$$\begin{aligned}
\frac{d}{dx}(4x^4 - 1)^3/7 &= \frac{3}{7}(4x^4 - 1)^{-4/7}\frac{d}{dx}(4x^4 - 1) \\
&= \frac{3}{7}(4x^4 - 1)^{-4/7}(16x^3) \\
&= \frac{3}{7}(16x^3)(4x^4 - 1)^{-4/7} \\
&= \boxed{\frac{48}{7}\frac{x^3}{(4x^4 - 1)^{4/7}}.}
\end{aligned}$$

3. Calculate $\dfrac{d}{dx}\tan^6(2x)$.

Solution: We must take the derivative of a function raised to a power. We'll use the Power Rule.

$$\begin{aligned}
\frac{d}{dx}\tan^6(2x) &= 6\tan^5(2x)\frac{d}{dx}\tan(2x) && \triangleleft \text{ Power Rule} \\
&= 6\tan^5(2x)\sec^2(2x)(2) && \triangleleft \text{ Trig. (3)} \\
&= \boxed{12\sec^2(2x)\tan^5(2x).}
\end{aligned}$$

Quiz/Solutions: Week 8.

No quiz due to test.

Quiz/Solutions: Week 9.

Quiz #6 is the take-home homework assignment over the Spring Break. Questions and solutions are available.

Quiz/Solutions: Week 10.

Quiz #7

1. Find the absolute extrema of the function $f(x) = x^3 - 12x + 1$ over the interval $[-3, 5]$.

Solution: We find the critical points of f,

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4).$$

It is clear from this calculation that f has two critical points:

$$\text{Critical Points: } x = \pm 2.$$

Note that both points lie in the given interval. Now we toss the end-points into the mix:

$$\text{Test Points: } -3, -2, 2, 5$$

and calculate f at each of these points.

x	$f(x)$	
-3	10	
-2	17	
2	-15	$\Longleftarrow \min f(x)$
5	66	$\Longleftarrow \max f(x)$

2. Find the intervals of increase and decrease of $f(x) = x^3(x-4)^4$.

Solution: We begin by obtaining the first derivative of f.

$$\begin{aligned} f'(x) &= 4x^3(x-4)^3 + 3x^2(x-4)^4 \\ &= x^2(x-4)^3[4x + 3(x-4)] \\ &= x^2(x-4)^3(7x-12) \end{aligned}$$

Having calculated f' and completely factored same, it is easy to use the *Sign Chart Method* to determine the intervals over which f' is positive and negative. The final results are

$$f \text{ is increasing over: } (-\infty, 12/7) \cup (4, +\infty)$$
$$f \text{ is decreasing over: } (12/7, 4)$$

Quiz/Solutions: Week 11.

No quiz because of Test #3 the same week.

Quiz/Solutions: Week 12.

No quiz this week.

Quiz/Solutions: Week 13.

Quiz #8

1. Calculate $\displaystyle\int 4x^3 - \frac{6}{x^2} + x^{3/2}\,dx.$

Solution: First prepare the integrand for the Power Rule, then apply the **Power Rule**:

$$\begin{aligned}\int 4x^3 - \frac{6}{x^2} + x^{3/2}\,dx &= \int 4x^3 - 6x^{-2} + x^{3/2}\,dx \\ &= 4\frac{x^4}{4} - 6\frac{x^{-1}}{-1} + \frac{x^{5/2}}{5/2} + C \\ &= \boxed{x^4 + \frac{6}{x} + \frac{2}{5}x^{5/2} + C.} \qquad \square\end{aligned}$$

2. Evaluate $\int_1^4 \sqrt{x} + x\,dx$.

Solution: First prepare integrand for the power rule, then apply same.

$$\begin{aligned}
\int_1^4 \sqrt{x} + x\,dx &= \int_1^4 x^{1/2} + x\,dx \\
&= \frac{2}{3}x^{3/2} + \frac{1}{2}x^2\Big|_1^4 \\
&= \frac{2}{3}(8-1) + \frac{1}{2}(16-1) \\
&= \frac{14}{3} + \frac{15}{2} \\
&= \boxed{\frac{73}{6}.} \quad \square
\end{aligned}$$

3. Evaluate $\displaystyle\int \left(x^3 - \frac{2}{x^3}\right)^2 dx$.

Solution: Multiply out the integrand and apply the power rule.

$$\begin{aligned}
\int \left(x^3 - \frac{2}{x^3}\right)^2 dx &= \int x^6 - 4 + 4x^{-6}\, dx \\
&= \frac{1}{7}x^7 - 4x + 4\frac{x^{-5}}{-5} + C \\
&= \boxed{\frac{1}{7}x^7 - 4x - \frac{4}{5x^5} + C.} \quad \square
\end{aligned}$$

Quiz/Solutions: Week 14.

Quiz/Solutions: Week 15.