THE UNIVERSITY OF AKRON
Mathematics and Computer Science

⇑

calculus 1
spring '96

Article: Summary of Lectures

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Summary of Lectures

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1. Week 1: 1/16/96 – 1/19/96

The Spring Semester, 1996, began on Tuesday, January 16, 1996, due to the *Martin Luther King, Jr.* Holiday.

1.1. Lecture 1

Text Reference: Chapter 0, §1

An overview of the course was given. *Calculus* is the study of *functions* and their properties. The four big topics of *Calculus* are

1. Functions;
2. Limits of Functions;
3. Continuity of Functions;
4. Differentiation of Functions;
5. Integration of Functions.

Throughout the semester we will study each of these topics in some detail. There are many *applications* to each of these topics; these also will be surveyed.

Limits is a the most fundamental concept of topics (2) – (5) above. The concepts of continuity, differentiation, and integration all depend on the *limit process*. This will be our first topic of study. (Following in introduction to functions.)

The five topics listed above are all extensively discussed in this tutorial. In the course of the semester, as I type up these summary lecture notes, I'll try to cross reference lecture topics with tutorial headings via the *hypertext link*.

Having dispatched all the usual bureaucratic duties, I began by introducing the notion of a function.

The definition was first discussed. See the discussion in the section entitled Functions: Basic Concepts for some of the comments made. This section is devoted to terminology, notation, and fundamental concepts.

1.2. Lecture 2

Text Reference: Chapter 0, §1

In this lecture, I discussed ways of thinking about functions: Models for Functions. These are abstract ways of thinking of functions. The *Venn Diagram* depiction of a function and the "input-output" model was mentioned.

It was then stressed that when a function is defined, the role of the letter used to refer to the independent variable (input value?) is *irrelevant*. For example, all the definitions define exactly the same function:

a. $f(x) = x^2$;
b. $f(t) = t^2$;
c. $f(s) = s^2$;
d. $f(W) = W^2$;

See the discussion The Independent Variable is a Dummy in the tutorials for more details.

Evaluation techniques were also discussed. Numerical evaluation does not usually give the student much problem (maybe):

$$f(x) = x^2 \implies f(2) = 4$$

When the quantity to be evaluated is itself an algebraic expression, then things take a turn for the worse. For example, suppose the function f is as above, what is the evaluation of the symbolisms: $f(x^2+1)$ or $f(1/\sqrt{w})$? The answers are

$$f(x^2+1) = (x^2+1)^2$$

and,

$$f(1/\sqrt{w}) = \frac{1}{w}.$$

The rule for making such evaluations is very simple. Suppose we have a function

$$g(x) = \frac{x}{x^2+1}. \tag{1}$$

The letter x is this context is used to describe the process of evaluating. The definition of g in (1) states that whatever g finds insides its parentheses, g takes that quantity and performs a series of calculations using that quantity according to the left-hand side of (1).

Thus,

$$g(w+1) = \frac{w+1}{(w+1)^2+1}. \tag{2}$$

Here, g has found the expression $w+1$ within its parentheses. The function g takes this expression captured inside it's parentheses, and constructs the left-hand side of (2) using the (1) *as a templet*!

This idea of "parentheses capturing" is quite general. For example, if f is another function then you should be able to evaluate the symbolism: $g(f(x))$; indeed,

$$g(f(x)) = \frac{f(x)}{[f(x)]^2+1}.$$

This is because $f(x)$ is captured inside the parentheses of g.

See the discussion on *The Argument of a Function* for additional details and examples.

Some discussion of the domain of a function was given. I noted that when defining a function one of the following occurs:

a. The domain of the function is completely specified at definition time; for example,

$$f(x) = \sqrt{x+1} \qquad x \geq 6.$$

b. The domain of the function is *not* specified at definition time; for example,

$$g(x) = \sqrt{x+1}.$$

In this case, the domain is taken to be the so-called natural domain. In this example, the natural domain is given by

$$\mathrm{Dom}(g) = [\,-1, \infty\,).$$

The natural domain is computed by identifying any (natural) restrictions on the calculation of the function: The function *must* evaluate to a real number. To major restriction commonly seen are

1. $\sqrt{x}$ is real provided $x \geq 0$;

and,

2. $\dfrac{a}{b}$ is real provided $b \neq 0$.

See the discussion on techniques of calculating the natural domain.

1.3. Lecture 3

Text Reference: Chapter 0, §1

I continued my discussion on domain calculations, then moved on to the discussion of the Algebra of Functions. Here, I talked about the fundamental arithmetic operations performed on functions: scalar multiplication, addition, multiplication, and division.

2. Week 2: 1/22/96 – 1/26/96

Let's begin Week 2.

2.1. Lecture 4

Text Reference: Chapter 0, §1

Composition of functions was first up. When we compose two functions together, we are taking the "output value" of one function and use it as the "input value" of the other function.

Another idea thrown out is that of un-composing functions.

Both composition and uncomposing are very important and fundamental concepts that are used throughout mathematics. Uncomposing you will see later in the semester as an important skill needed to differentiate complex functions.

2.2. Lecture 5

Text Reference: Chapter 0, §2; Chapter 1, §1

A brief discussion of the affects on the graph of a function under common transformations: Shifting and Rescaling. The discussion includes horizontal shifting, vertical shifting, and rescaling. Also discussed were the affects on a graph by taking the absolute value.

Passing on now with great speed, we move to the next chapter!

An introduction to the notion of limits was started. Before class ended, I managed to begin discussion of the *Fundamental Problem of Differential Calculus*. Checkout this reference for some of the graphics that I was drawing on the board — including an animated picture!

2.3. Lecture 6

Text Reference: Chapter 1, §1

Motivating the concept of limit. I discussed the *Fundamental Problem of Differential Calculus*. Checkout this reference for some of the graphics that I was drawing on the board — including an animated picture!

The Basic Idea: Given a function $f(x)$ and a number a. We are interested in the *behaviour* of $f(x)$ as x gets "closer and closer" to a. That is, as x gets closer and closer to the number a, we want to see if there is a corresponding trend in the values of $f(x)$. When we write

$$L = \lim_{x \to a} f(x)$$

we mean, in rough terms, that as x gets "closer and closer" to a, $f(x)$ gets "closer and closer" to the number L.

In addition to the tangent line, this section of the book also discusses, as motivation to the concept of limit, the notion of instantaneous velocity. A brief discussion of instantaneous velocity was given. More on the morrow — meanwhile, the references to these topics can be explored.

2.4. Lecture 7

Text Reference: Chapter 1, §1–§2

Discussion of velocity continued.

Moving now into §2, a discussion of limit included

$$\lim_{x \to a} f(x) = L$$

means,

> "As x gets closer and closer to a, $f(x)$ gets closer and closer to L."

Some examples followed.

3. Week 3: 1/29/96 – 2/2/96

3.1. Lecture 8

Text Reference: Chapter 1, §2

I outlined a number of limit concepts:

a. (Bi-directional) Limit.
b. (Uni-directional, or) One-Sided Limits.
c. Relationship between (a) and (b).
d. Infinite limits.

The bi-directional limit is the limit that we have been discussing throughout. One-sided limits are useful when dealing with function that have piecewise definitions.

It is important to use the proper notation with dealing with limits — being literate is an important aspect of understanding the concepts.

Consider the function:

$$f(x) = \begin{cases} x^2 & x < 1 \\ 2x + 3 & x \geq 1 \end{cases}$$

As discussed in class we have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1 \qquad \triangleleft \text{ left-hand limit}$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} 2x + 3 = 5 \qquad \triangleleft \text{ right-hand limit}$$

One-sided limits can be used to analyze two-sided (bi-directional) limits. The definitive theorem gives the relationship:

$\lim_{x \to a} f(x) = L$ if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Infinite limits are useful as the formal device for describing *vertical asymptotes*.

$$\lim_{x\to 1} \frac{x}{(x-1)^2} = +\infty$$

describes a vertical asymptote. This two-sided limit indicate that the graph of the function "zooms off to infinity" to the left and to the right of $x = 1$.

One sided infinite limits refine the description of the vertical asymptote:

$$\lim_{x\to 1^-} \frac{x}{(x-1)^3} = -\infty$$
$$\lim_{x\to 1^+} \frac{x}{(x-1)^3} = +\infty$$

describes a vertical asymptote take plunges to $-\infty$ as x approaches 1 from the left, and zooms off to $+\infty$ as x approaches 1 from the right.

3.2. Lecture 9

Text Reference: Chapter 1, §3

The whole period was taken up discussing the Algebra of Limits. Basically, these state that limits of functions behave exactly as you would expect them to behave.

There are exceptions. It was pointed out in class that there is an underlying assumption that the limits of the individual functions exist and are finite. I presented this example for your consideration:

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x} \right) = 0$$

whereas,

$$\lim_{x \to 0} \frac{1}{x} - \lim_{x \to 0} \frac{1}{x} \quad \text{d.n.e.}$$

This shows that, in general, it is *not true* that

$$\lim_{x \to a} [f(x) \pm g(x)] \neq \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

unless it is known that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ *both exist and are finite.*

Most interesting is the case of quotients of functions: The limit of a quotient is the quotient of the limits provided the limit of the denominator is nonzero.

Example 3.1. Calculate $\displaystyle\lim_{x\to 2} \frac{x^2-2x+5}{x^2+3x+1}$.

Solution: We proceed along standard line of thought.

$$\lim_{x\to 2} \frac{x^2-2x+5}{x^2+3x+1} = \frac{\lim\limits_{x\to 2}(x^2-2x+5)}{\lim\limits_{x\to 2}(x^2+3x+1)} \qquad \triangleleft \text{ quotient rule}$$

$$= \frac{2^2-2(2)+5}{2^2+3(2)+1}$$

$$= \boxed{\frac{5}{11}.}$$

It is important to note that the limit of the denominator was 11, which is nonzero. This observation justifies the use of the quotient rule for limits.

This is a *Skill Level 0* problem. Example 3.1. ■

That example was easy enough. But they are not all easy. Take a look at this example.

Example 3.2. (Skill Level 1) Calculate $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2}$

Solution: A preliminary analysis shows that the denominator goes to zero as $x \to 0$. What saves us is that the numerator goes to zero also! Here's the details.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} && \triangleleft \text{ factor!} \\
&= \lim_{x\to 2} (x+2) && \triangleleft \text{ cancel!} \\
&= \boxed{4} && \triangleleft \text{ done!}
\end{aligned}$$

Here we are taking the limit of a quotient of two functions. The denominator is going to zero, so the standard quotient rule does *not apply*. When in this situation, the rule of thumb would be to look for factor common to both numerator and denominator. In this case it was $x - 2$. Once we canceled the offending factors, we were back to *Skill Level 0*. Example 3.2. ■

More examples and exercises can be found in the book (Ch. 1, §3), or the tutorials

3.3. Lecture 10

Text Reference: Chapter 1, §3

Quiz #2 was announced for friday covering the ideas and techniques in §2–§3 of Chapter 1.

The period was spent doing representative examples of limit problems. All the examples involve ratios of functions that are tending to zero. In this situation, we utilize the *Empirical Observation*: Perform algebraic manipulation with the goal of cancelling out the offending factors. (See yesterday's lecture for links to examples in this tutorial.)

The *Squeeze Theorem* was also discussed. This technique can be described roughly as follows:

Problem: Show $\lim_{x \to a} h(x) = L$.

The Method: Construct two function f and g such that

1. $f(x) \leq h(x) \leq g(x)$ for x "close" to a; and
2. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$.

Then

$$\lim_{x \to a} h(x) = L.$$

The functions f and g are constructed by the user — that's you!

The example done in class was $\lim_{x \to 0} x \sin(1/x)$. The target function is $h(x) = x \sin(1/x)$. It is well-known (by you, I hope) that for *any* number z,

$$-1 \leq \sin(z) \leq 1.$$

Now, thinking of the number z as $1/x$ we obtain

$$-1 \leq \sin(\frac{1}{x}) \leq 1,$$

or,

$$|\sin(\frac{1}{x})| \leq 1. \tag{1}$$

Now, take the inequality (1) and multiply both sides by the nonnegative quantity $|x|$ to obtain

$$|x \sin(\frac{1}{x})| \leq |x|. \tag{2}$$

Finally, you'll remember from algebra that

$$|a| \leq b \iff -b \leq a \leq b \tag{3}$$

Finally, taking (2) and applying (3), we obtain:

$$\boxed{-|x| \leq x \sin(\frac{1}{x}) \leq |x|} \tag{4}$$

This is the desired inequality. Now that we have our given function "squeezed" between two others, we need only observe

$$\lim_{x \to 0} |x| = 0 = \lim_{x \to 0} (-|x|) \tag{5}$$

Line (4) corresponds to (1) and (5) corresponds to (2). We are justified in concluding, by the *Squeeze Theorem*:

$$\boxed{\lim_{x \to 0} x \sin(\frac{1}{x}) = 0.}$$

'Nuff said!

3.4. Lecture 11

Text Reference: Chapter 1, §4

An introduction to the technical definition of limit was given. We started with the "pedestrian definition" and refined it. Out of necessity, we had to formulate the meaning of the word "close."

Close. Let A and B be numbers and $\delta > 0$ a positive number. We say that B is close to A (or, A is close to B) provided

$$|A - B| < \delta.$$

That is, the distance ($|A - B|$) between A and B is less than a preselected number δ. The number δ "controls" the meaning of the word "close." If we choose $\delta = 1/10$, then closeness means "within $1/10^{\text{th}}$." If we choose δ even smaller, say $\delta = 1/100$, then closeness now means "within $1/100^{\text{dreth}}$."

Definition. We say that

$$\lim_{x \to a} f(x) = L,$$

provided, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Paraphrased, we can say: For any meaning of the word "close to L" we want to make (that's the "for any $\epsilon > 0$" part), we can get that close by choosing ("there is a $\delta > 0$") a meaning of the word "close to a" such that if x is that close ("$0 < |x - a| < \delta$") then $f(x)$ is close to L ("$|f(x) - L| < \epsilon$").

An simple example was presented to illustrate the definition. A complete presentation of the technical definition of limit with many examples and exercise is given in the section entitled Working with the Definitions.

4. Week 4: 2/5/96 – 2/9/96

4.1. Lecture 12

Text Reference: Chapter 1, §4

Continuing the discussion on limits for yesterday, I presented several examples of using the definition to prove the correctness of our intuition.

Here is a representative example — not necessarily the example I did in class.

Example 4.1. Prove $\lim_{x \to -2} (4x - 3) = -11$.

Solution: Recalling the definition, we begin by letting $\epsilon > 0$ be given. We need to find a number $\delta > 0$ such that

$$0 < |x - (-2)| < \delta \implies |(4x - 3) - (-11)| < \epsilon$$

or,

$$0 < |x + 2| < \delta \implies |4x + 8| < \epsilon \qquad (1)$$

Side Calculations: (You don't see this!) Basically, we want

$$|(4x - 3) - (-11)| < \epsilon. \tag{2}$$

We are interested in identifying those values of x that will cause (2) to be true. Let's see . . .

$$\begin{aligned} |(4x - 3) - (-11)| < \epsilon &\iff |4x + 8| < \epsilon \\ &\iff 4|x + 2| < \epsilon \\ &\iff |x + 2| < \frac{\epsilon}{4} \qquad (3) \end{aligned}$$

Comparing (3) to the left-hand side of (1), and after many hours of meditation, we "see" that $\delta = \epsilon/4$.

Now see this: Choose $\delta = \dfrac{\epsilon}{4}$, then

$$\begin{aligned} 0 < |x+2| < \delta &\implies |x+2| < \frac{\epsilon}{4} \\ &\implies 4|x+2| < \epsilon \\ &\implies |4x+8| < \epsilon \\ &\implies |(4x-3)-(-11)| < \epsilon. \end{aligned}$$

Thus, we have shown that

$$0 < |x+2| < \delta \implies |(4x-3)-(-11)| < \epsilon.$$

This proves $\lim_{x \to -2}(4x-3) = -11$ from the definition.

Example 4.1. ■

4.2. Lecture 13

Text Reference: Chapter 1, §5

Definiton. A function f is continuous at $x = a$ provided

a. $a \in \text{Dom}(f)$;
b. $\lim_{x \to a} f(x)$ exists and is finite;
c. $\lim_{x \to a} f(x) = f(a)$.

A complete discussion of this topic is given in the article entitled Continuous Functions.

As was mentioned in class, *continuity* corresponds to limit problems of *Skill Level* 0. For example, if we are asked to calculate the limit problem:

$$\lim_{x \to 2} \frac{\sqrt{x^2 + 3}}{3x + 5},$$

we would realize that the limit of the denominator is nonzero, and, consequently, the limit problem is *of Skill Level* 0. Indeed,

$$\lim_{x \to 2} \frac{\sqrt{x^2+3}}{3x+5} = \frac{\sqrt{2^2+3}}{3(2)+5} = \boxed{\frac{\sqrt{7}}{11}.}$$

That is, the limit can be calculated by evaluating underlying function at $x = 2$.

The technical reason that we can evaluate this limit in this way is because the underlying function

$$f(x) = \frac{\sqrt{x^2+3}}{3x+5}$$

is *continuous* at $x = 2$.

The evaluation of the limit of continuous function, then, is quite simple — *provided you* know *the function is continuous*! This brings us around to a discussion of common continuous functions.

In class, the algebra of continuous functions was discussed, and, as a result, classes of continuous functions were identified: polynomials, rational functions, and algeraic functions.

4.3. Lecture 14

Text Reference: Chapter 1, §5

The intermediate value theorem was discussed. Geometrically, a continuous function is a function whose graph can be drawn without picking up your pencil. (This last statement is not really true, but for all the functions we shall encounter, the description is accurate.)

A *Pedestrian Description* of the **Intermediate Value Theorem** can be stated as follows: "The graph of a continuous function attains all altitudes *between* any two given points on its graph."

One of the important applications to this theorem is to *root hunting*! Let $f(x)$ be a given function. A *root* of f is any number $a \in \mathrm{Dom}(f)$ such that $f(a) = 0$; or, in other words, a root of f is any solution to the equation $f(x) = 0$. Graphically, roots of f are the x-intercepts: The points at which the graph of the function crosses the x-axis.

In mathematics, it is not unusual to want to solve the equation:

$$f(x) = 0. \tag{4}$$

It would be a waste of time to try to solve (4) if there are no solutions; therefore, the first step towards solving an equation is to determine whether there are any solutions at all! The **Intermediate Value Theorem** can be very useful in this regard.

For example, consider the equation,

$$3x^3 - 4x^2 + 8x - 1 = 0 \tag{5}$$

Here, we want to find the roots of the *continuous function*

$$f(x) = 3x^3 - 4x^2 + 8x - 1.$$

By examining the values of this function, we come across the following two calculations:

$$f(0) = -1 \text{ and } f(1) = 6.$$

That is, f is negative at $x = 0$ and f is positive at $x = 1$. By the **Intermediate Value Theorem**, the function f must attain all altitudes in between -1 and 6. Here's the critical observation: The number 0 is between -1 and 6! Somewhere between 0 and 1, f *must* take on a value of 0.

Thus, there is a number $c \in (\,0, 1\,)$ such that $f(c) = 0$. This means that there is a root of f somewhere between 0 and 1. Now that we know that there is a root between 0 and 1, we can gather our resources and set off to find it! (Left to the reader, good luck!)

O.k., O.k., I've calculated the root on my computer and the value is

$$\boxed{x_{\text{root}} = 0.1329574493}$$

and furthermore, this is the only root!

4.4. Lecture 15

Text Reference: Chapter 1, §6

Discussion of a new representation the slope of the tangent line, instantaneous velocity was discussed.

Tangent Line Slope: Let $y = f(x)$ be given, and let $a \in \text{Dom}(f)$, then

$$m_{\tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \triangleleft \text{ from §1.1} \tag{6}$$

$$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad \triangleleft \text{ from §1.6} \tag{7}$$

The representation given in (7) is often much simpler to use in calculations of particular problems than (6). Let me illustrate that point with an example.

Example 4.2. Calculate the $m_{\tan}$ for $f(x) = x^3$ at $x = a$.

Solution: We solve this twice, first using (7), then using (6).

First Try:

$$
\begin{aligned}
m_{\tan} &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{(a+h)^3-a^3}{h} \\
&= \lim_{h\to 0} \frac{a^3+3a^2h+3ah^2+h^3-a^3}{h} \\
&= \lim_{h\to 0} \frac{3a^2h+3ah^2+h^3}{h} \\
&= \lim_{h\to 0} (3a^2+3ah+h^2) \qquad \triangleleft \text{ cancel `h'!} \\
&= 3a^2
\end{aligned}
$$

Thus, $\boxed{m_{\tan} = 3a^2}$.

Those calculations were straight forward. Must manipulations were multiplying out quantities and dividing out the h.

Second Try:

$$
\begin{aligned}
m_{\tan} &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{x^3 - a^3}{x - a} \qquad (8) \\
&= \lim_{x \to a} \frac{(x - a)(x^2 + xa + a^2)}{x - a} \\
&= \lim_{x \to a} (x^2 + xa + a^2) \qquad \triangleleft \text{ cancel!} \\
&= a^2 + aa + a^2 \\
&= 3a^2
\end{aligned}
$$

Line (8) is the critical. Here, I had to be able to factor a third degree polynomial — potentially, a difficult task. Compare the problem of factoring in this method, with the previous method. There, no factorization was needed, just expanding expressions.

As a general rule, the "h" form of the definition of $m_{\tan}$ is usually easier to use.

Example 4.2. ■

The notion of *instantaneous velocity* was also discussed. A similar modification of the original formula was developed:

$$v(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} \qquad \triangleleft \text{ from §1.1}$$

$$v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \qquad \triangleleft \text{ from §1.6}$$

Again, the second formulation is often simpler to work with.

5. Week 5: 2/12/96 – 2/17/96

5.1. Lecture 16

Text Reference: Chapter 2, §1

The formal definition of *derivative* was given. A general reference in the tutorials is the article on Differentiation; in particular, the subsection of that article entitled The Definition of Derivative contains many examples of derivative calculations at the definition level.

Let $y = f(x)$ be a function and $a \in \mathrm{Dom}(f)$ an interior point of the domain. The derivative function is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

provided the limit exists and is finite.

The following example is similar to the one done in class.

Example 5.1. For $f(x) = x^3$, calculate $f'(x)$.

Solution: We build up the expression on the right-hand side of (1).

$$\begin{aligned}
f(x) &= x^3 \\
f(x+h) &= (x+h)^3 \\
f(x+h) - f(x) &= (x+h)^3 - x^3 \\
&= x^3 + 3x^2h + 3xh^2 + h^3 - h^3 \\
&= 3x^2h + 3xh^2 + h^3 \\
&= h(3x^2 + 3xh + h^2) \\
\frac{f(x+h) - f(x)}{h} &= 3x^3 + 3xh + h^2 \qquad (2)
\end{aligned}$$

and finally,

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \triangleleft \text{ from (1)} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 && \triangleleft \text{ from (2)} \\
&= 3x^2.
\end{aligned}$$

Thus, $\boxed{f'(x) = 3x^2}$.

Example 5.1. ■

It was *emphasized* that the derivative function, such as $f'(x) = 3x^2$ as in the previous example, contains **all** tangent line information. If you want to access a particular piece of information, you must take the derivative function, and evaluate it at a particular value of x. For example,

Example 5.2. For $f(x) = x^3$, find the equation of the line tangent to the graph of f at $x = 2$.

Solution: From Example 5.1, $f'(x) = 3x^2$. We are interested in the case $x = 2$, thus, $f'(2) = 12$. The interpretation of this is that $f'(2) = 12$ is *the slope of the line tangent to the graph of f at $x = 2$*. The tangent line in question passes through the point

$$(x_0, y_0) = (2, f(2)) = (2, 8)$$

and

$$m_{\tan} = f'(2) = 12.$$

Thus, the equation of the line tangent to the graph of f at $x = 2$ is,

$$\boxed{y - 8 = 12(x - 2)}$$

or,

$$\boxed{y = 12x - 16.}$$

Example 5.2. ■

5.2. Lecture 17

Text Reference: Chapter 2, §1–§2

I introduced some standard notation for differentiation. An exhaustive discussion of notation is presented in the section entitled Derivative Notation. In this section, the prime notation and the Leibniz notation are covered.

I then began §2 of Chapter 2 in the text by talking about differentiation by formula. See Some Basic Differentiation Rule for an on-line discussion. In class, I covered the formula

$$\frac{d\,c}{dx} = 0 \qquad \triangleleft \text{ Der. Const.}$$

Next I presented the Power Rule:

$$\frac{d}{dx} x^n = n x^{n-1} \qquad n = 1, 2, 3, \ldots$$

Finally, as time ran out (I should have closed the door on time!), the homogeneity property and the additive rule for differentiation were stated and illustrated.

5.3. Lecture 18

Text Reference: Chapter 2, §1

Test #1 Today, no lecture!

5.4. Lecture 19

Text Reference: Chapter 2, §2

Continuing discussion of the rules for differentiation: the product rule and the quotient rule. Several examples were presented.

The importance of good notation was emphasized. While doing problems at home, use good and correct notation. Slowly and methodically work through a differentiation problem. Do it write the first time.

Here is an example of good notation:

Example 5.3. Calculate $\dfrac{d}{dx}\dfrac{x}{x^2+1}$.

Solution: We are asked to calculate the derivative of a *quotient*. We apply the quotient rule:

$$\frac{d}{dx}\frac{x}{x^2+1} = \frac{(x^2+1)\dfrac{dx}{dx} - x\dfrac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

◁ verbalize!

$$= \frac{(x^2+1) - x(2x)}{(x^2+1)^2} \qquad \triangleleft \text{ Power Rule}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

Example 5.3. ■

Finally, the power rule can be extended to arbitrary numerical exponents. See the section on General Power Rule. Here's a couple of simple examples to illustrate, other examples can be seen in the section General Power Rule in the tutorials.

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$$

$$\frac{d}{dx}x^{-4/5} = -\frac{4}{5}x^{-9/5}$$

Finally, the period ended, and I handed out the Weekend Assignment.

6. Week 6: 2/19/96 – 2/23/96

6.1. Lecture 20

Text Reference: Chapter 2, §2

Handed back the test. The test average was 75% — well done class!

More examples of the differentiation rules were presented. It was emphasized that utilizing proper notation of paramount importance: Use the Leibniz notation to help you through a complex problem.

Another important point was made. When looking at a differentiation problem, first consider whether you should manipulate the function to be differentiated for the purpose of simplifying it for differentiation. An example was presented to make the point.

Problem. Differentiate $\dfrac{d}{dx}\dfrac{x}{x+\frac{c}{x}}$. You can differentiate this function immediately, or you can simplify it *before* differentiating. Thus,

$$\frac{d}{dx}\frac{x}{x+\frac{c}{x}} = \frac{d}{dx}\frac{x^2}{x^2+c} \tag{1}$$

The differentiation problem on the right-hand side of (1) is much easier to calculate than the differentiation problem on the left-hand side of (1).

6.2. Lecture 21

Another Holiday!

6.3. Lecture 22

Text Reference: Chapter 2, §4

A quick review of the trigonometric functions was given. I began to develop some basic limit properties of the trig. functions. See the section entitled *Trigonometric Limits* in the tutorials.

6.4. Lecture 23

Text Reference: Chapter 2, §4

Finished proving the basic limit results needed to develop the differentiation rules for the trignometric functions. These are

1. $\lim_{x \to 0} \sin(x) = 0$.
2. $\lim_{x \to 0} \cos(x) = 1$.
3. It was noted that (1) and (2) imply that the sine and cosine functions are everywhere continuous.
4. $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$.
5. $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x} = 0$.

Some basic limit techniques were demonstrated as well:

$$\lim_{x\to 0} \frac{\sin(3x)}{x} = 3 \lim_{x\to 0} \frac{\sin(3x)}{3x} = 3(1) = 3$$

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{\cos(x)} = 1$$

$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)} = \frac{2}{3} \lim_{x\to 0} \frac{\sin(2x)/(2x)}{\sin(3x)/(3x)} = \frac{2}{3}$$

All of the above results utilize the above enumerated limit results.

Finally, Quiz #4 was given.

7. Week 7: 2/26/96 – 3/1/96

7.1. Lecture 24

Text Reference: Chapter 2, §4-§5

See the section entitled The Trigonometric Functions for an on-line discussion of the development of the derivative formulas along with many examples.

Started off by deriving all the differentiation formulas. The difference quotient for the function $\sin(x)$ is

$$\frac{\sin(x+h)-\sin(x)}{h} = \cos(x)\left(\frac{\sin(h)}{h}\right) + \sin(x)\left(\frac{\cos(h)-1}{h}\right) \tag{1}$$

The derivative of $\sin(x)$ is the limit of the above expression as h goes to zero. Since

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0 \tag{2}$$

It is now easy to see from (1) and (2) that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Similarly, we can argue that

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

The other trig functions are quotients of sines and cosines, and so their derivatives are easily computed. See the Trig. Table for a complete listing — and the discussion following.

The Chain Rule. This is the rule that allows us to compute the derivative of a function that is realizable as the composition of two other differentiable functions.

Suppose $F(x)$ is a function to be differentiated. Suppose we realize that this F is the composition of two other functions: $F = f \circ g$. Assuming these functions are themselves differentiable, the Chain Rule basically states:

$$F'(x) = (f \circ g)'(x) = f'(g(x))g'(x).$$

At the end of the period, a generalized power rule was introduced:

$$\frac{d}{dx}[g(x)]^r = r[g(x)]^{r-1}g'(x)$$

and illustrated. See Power Rule Revisited for more examples and discussion.

7.2. Lecture 25

Text Reference: Chapter 2, §5-§6

Continuation of the chain rule. The Liebniz Notation is discussed.

With the notation of Leibniz, the old differentiation formulas can be generalized. Let u be a differentiable function of x.

1. (Power Rule) $\dfrac{d}{dx}u^r = ru^{r-1}\dfrac{du}{dx}$.

2. $\dfrac{d}{dx}\sin(u) = \cos(u)\dfrac{du}{dx}$.
3. $\dfrac{d}{dx}\cos(u) = -\sin(u)\dfrac{du}{dx}$.
4. $\dfrac{d}{dx}\tan(u) = \sec^2(u)\dfrac{du}{dx}$.
5. $\dfrac{d}{dx}\cot(u) = -\csc^2(u)\dfrac{du}{dx}$.
6. $\dfrac{d}{dx}\sec(u) = \sec(u)\tan(u)\dfrac{du}{dx}$.
7. $\dfrac{d}{dx}\csc(u) = -\csc(u)\cot(u)\dfrac{du}{dx}$.

Several examples of the use of these formula were given.

7.3. Lecture 26

Text Reference: Chapter 2, §6

A quiz was announced for friday convering the topics of §§2.4-2.5. Also, I postulated the existence of Test #2 for friday of next week.

The Method of Implicit Differentiation was discussed. This method is used to calculate the slope of a tangent line for curves described by an equation: $F(x, y) = c$. See the above hyperlink for complete details/examples.

7.4. Lecture 27

Text Reference: Chapter 2, §7-§8

The topic of Higher Order Derivatives was taken up. Computing higher order derivatives using explicit differentiation techniques as well as implicit differentiation techniques.

Related Rate Problems. This is the subject of §2.8. Here is a bear bones description of the idea.

Suppose we are studing some physical system involving variables x and y, which vary in time t. Suppose there is a physical law that relates x and y; symbolically,

$$F(x, y) = c, \tag{3}$$

where x and y are considered function of t.

Given that x and y are related by (3), and both x and y are changing in time, we ask the question: What is the relationship between dx/dt and dy/dt.

The idea to apply the method of implicit differentiation to (3),

$$\frac{d}{dt}F(x,y) = 0.$$

Typically, after the rules for differentiation are applied, we get an equation of the general form:

$$G(x,y,\frac{dx}{dt},\frac{dy}{dt}) = 0. \tag{4}$$

Given x and y are related by (3), then (4) tells how dx/dt and dy/dt are related.

Now here's the point of this exercise. If we know how fast one of the two variables is changing (i.e. if we know dx/dt, say), then we can figure out how fast the other variable is changing.

Quiz #5 was given . . . the solutions to which are posted.

8. Week 8: 3/4/96 – 3/8/96

8.1. Lecture 28

Text Reference: Chapter 2, §8

A couple of representative examples of *Related Rate Problems* were presented in class.

Text Reference: Chapter 2, §8-§9

More related rate problems were done. Introduction to the concept of the differential. The definition of a differential is given as

$$y = f(x)$$
$$dy = f'(x)\,dx$$

where, dx is a new independent variable, called the *differential of* x and dy is a new dependent variable, called the *differential of* y. The uses of the notion of differential are several:

1. Notational;
2. Geometric Interpretation;
3. dy as an approximation of Δy.

All of these are discussed extensively in the section referenced earlier on differentials.

8.2. Lecture 29

Text Reference: Chapter 2, §9

Differentials continued.

8.3. Lecture 30

Text Reference: Chapter 2, §1

Test #2.

9. Week 9: 3/11/96 – 3/15/96

9.1. Lecture 31

Text Reference: Chapter 2, §9

Differentials continued. The primary focus was on *Linear Approximations*. For a given function $y = f(x)$ and a given point $x = a$, the linear approximation of f at $x = a$ is given by

$$\boxed{L(x) = f(a) + f'(a)(x - a),} \tag{1}$$

this is nothing more then the equation of the tangent line.

For example, consider the function $f(x) = \sqrt{x+1}$ at $x = 3$. Then

$$f(x) = \sqrt{x+1}$$
$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

Specializing this to $x = 3$ we obtain,

$$f(a) = f(3) = \sqrt{4} = 2$$
$$f'(a) = f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The linear approximation of f at $x = 3$ is

$$L(x) = 2 + \frac{1}{4}x - 3$$

This means

$$\sqrt{1+x} \approx 2 + \frac{1}{4}x - 3 \qquad \text{near } x = 3.$$

For example, take $x = 3.1$, then

$$\sqrt{4.1} \approx 2 + \frac{1}{40} = 2.025.$$

A calculator calculation will show that this is a fairly accurate approximation.

9.2. Lecture 32

Text Reference: Chapter 2, §9-§10

Another example of linear approximation was presented, then I moved on to *quadratic approximation.* Quadratic approximation attempts to approximate a function $f(x)$ near a point $x = a$ using a polynomial of degree 2.

Problem: Find a polynomial $Q(x)$ that satisfies the following conditions:

1. Q is a polynomial of degree 2;
2. $Q(a) = f(a)$;
3. $Q'(a) = f'(a)$;
4. $Q''(a) = f''(a)$.

A polynomial of degree 2 can always be written in the form

$$Q(x) = A + B(x - a) + C(x - a)^2$$

this formulation makes it easy to figure out the values of the coefficients A, B, and C so that the resulting polynomial satisfies the 4 conditions above. In class, I showed that

$$A = f(a) \qquad B = f'(a) \qquad C = \frac{f''(a)}{2}.$$

Thus the quadratic approximation of f at $x = a$ is given by

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.} \tag{2}$$

A quick example of the use of the formula followed.

Suppose we are interested in estimating values of the function $f(x) = \sqrt{x}$ near $x = 1$. Let's calculate the linear and quadradic approximations.

Preliminary Calculations:

$$f(x) = x^{1/2} \qquad f(1) = 1$$

$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(x) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad f''(x) = -\frac{1}{4}$$

Therefore,

$$L(x) = 1 + \frac{1}{2}(x-1) \qquad \triangleleft \text{ from } (1)$$

$$Q(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-2)^2 \qquad \triangleleft \text{ from } (2)$$

Thus we have,

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) \qquad \text{near } x = 1$$

$$\sqrt{x} \approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-2)^2 \qquad \text{near } x = 1$$

Usually the second approximation is better.

An introduction to Newton's Method was given.

9.3. Lecture 33

Text Reference: Chapter 2, §10

Newton's Method continued.

Problem: Solve $f(x) = 0$.

Solution: Setup the *Newton Recursion Formula*

$$x_0 = \text{Initial Guess}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and make the calculations.

9.4. Lecture 34

Text Reference: Chapter 3, §1

A discussion of optimization problems was presented. In particular, a very useful and standard technique for finding absolute extrema we given.

Problem. Let f be a *continuous* function defined on the closed and bounded interval $[\,a, b\,]$. Find the absolute maximum and minimum values of f.

The Method.

1. Find the critical points of f over $[\,a, b\,]$: these are the numbers, x, at which $f'(x) = 0$ or $f'(x)$ does not exist. Let me represent these numbers symbolically:

$$x_1, x_2, x_3, \ldots, x_n.$$

2. Include the endpoints a and b is the above list:

$$a, x_1, x_2, x_3, \ldots, x_n, b. \tag{3}$$

3. Calculate the value of f at each of the numbers in the list (3).

x	$f(x)$
a	$f(a)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
$\vdots$	$\vdots$
x_n	$f(x_n)$
b	$f(b)$

The absolute maximum is the largest number in the right column and the absolute minimum is the smallest number in the right column.

10. Week 10: 3/25/96 – 3/29/96

10.1. Lecture 35

Text Reference: Chapter 3, §2–§3

Today was devoted to the development of general theory: ROLLE'S THEOREM and THE MEAN VALUE THEOREM. The second theorem forms the basis for many of the techniques used in this chapter as well as subsequent chapters … throughout the entire calculus series.

The important consequences of this theorem as developed in class are

1. If f and g are functions such that $f'(x) = g'(x)$ for all x in an interval $[a, b]$, then there must be a constant C such that

$$f(x) = g(x) + C,$$

   this means that the graph of f is a vertical shifting of the graph of g.
2. TESTING FOR MONOTONICITY Let f be a function that is differentiable over an interval I.

   (a) If $f' > 0$ over an interval I then f is monotone increasing over that interval.
   (b) If $f' < 0$ over an interval I then f is monotone decreasing over that interval.
3. The First Derivative Test. Let c be a critical number of the function f.
   (a) Suppose $f' > 0$ to the left of c and $f' < 0$ to the right of c. Then, f has a local maximum at c.
   (b) Suppose $f' < 0$ to the left of c and $f' > 0$ to the right of c. Then, f has a local minimum at c.

10.2. Lecture 36

Text Reference: Chapter 3, §3

The hour was spent utilizing the techniques enumerated in the list yesterday.

Given the function $f(x) = x^3 - 3x^2 + 1$, all first derivative information was extracted. Similarly, the first derivative information was extracted from the function $f(x) = x(1-x)^{2/5}$.

10.3. Lecture 37

Text Reference: Chapter 3, §4

Just as the first derivative contains certain information about the function (critical points, classification of same, and intervals of monotonicity); so too the second derivative contains information. The second derivative contains *concavity* information.

TESTS FOR CONVAVITY. Let f be a function having a second derivative over an interval I.

1. If $f'' < 0$ over the interval I, the graph of f is concave down.
2. If $f'' > 0$ over the interval I, the graph of f is concave up.

This testing procedure way used to analyze the concavity of the function $f(x) = x^3 = 3x^2 + 1$.

10.4. Lecture 38

Text Reference: Chapter 3, §5

Limits at infinity were discussed. These kinds of limits, when finite, are interpreted as *horizontal asymptotes*. Many “tricks of the trade” were demonstrated.

11. Week 11: 4/1/96 – 4/5/96

11.1. Lecture 39

Text Reference: Chapter 3, §6

General topics in curve sketching were discussed.

Test #3 was announced for next monday.

11.2. Lecture 40

Text Reference: Chapter 3, §8

Applied Maximum and Minimum Problems discussed.

11.3. Lecture 41

Text Reference: Chapter 3, §8

Applied Maximum and Minimum Problems — day 2.

11.4. Lecture 42

Text Reference: Chapter 3, §10

Introduction to the topic of *antiderivatives*.

12. Week 12: 4/8/96 – 4/12/96

12.1. Lecture 43

Test #3— no "action today."

12.2. Lecture 44

Text Reference: Chapter 3, §10, Chapter 4, §1

Antiderivatives concluded and introduction to summation notation.

12.3. Lecture 45

Text Reference: Chapter 4, §2

The area problem was discussed. This problem is used to illustrate the constructive nature of the definition of the definite integral (§4). The elaborate process of subdividing the interval into subintervals, choosing a point from each interval and forming *Riemann Sums* is typical of applications to the definite integral.

I reference A Little Problem with Area in the tutorial as well as the section entitled The Construction: A Morass of Notation. These two articles cover my comments today most adequetly.

12.4. Lecture 46

Text Reference: Chapter 4, §3

The formal definition of the definite integral was taken up. See the section The Definite Integral: The Final Definition for more details.

A question of existance of the definite integral is taken up (briefly) in the section entitled The Existence of the Definite Integral. The various properties of the definite integral was begun.

13. Week 13: 4/15/96 – 4/19/96

13.1. Lecture 47

Text Reference: Chapter 4, §3

Finished discussing the properties.

Turning to the problem of evaluating a definite integral, supporting theory was presented: THE FUNDAMENTAL THEOREM OF CALCULUS, PART I and THE FUNDAMENTAL THEOREM OF CALCULUS, PART II.

Elementary Examples were given.

13.2. Lecture 48

Text Reference: Chapter 4, §4

Attention was turned to the evalulation of the definite integral and indefinite integaral. Many examples were given.

Indefinite Integrals: Within the tutorials, a general discussion on the indefinite integral is given. Specific Formulas and General Formulas are also discussed. Many examples are given in the above references.

13.3. Lecture 49

Text Reference: Chapter 4, §5 The Technique of Substitution was developed, followed by a few illustrative examples. See the section entitled Learning the Technique of Substitution.

13.4. Lecture 50

Text Reference: Chapter 4, §5

When using the technique of substitution with the definition integral, there is an additional step involved: Calculating the limits of integration. See examples and exercises in the above reference.

Concerning the technique of substitution, I discern two attitudes that I find useful: Formula Checking and True Substitution of Variables. These two points of view may be helpful to you.

Finally, let me mention the section on Strategies for Integration. Here, I give a discussion some thinking processes needed to successfully solve problems in integration.

Quiz #2[3].

14. Week 14: 4/22/96 – 2/26/96

14.1. Lecture 51

Text Reference: Chapter 5, §1

Area between two curves was discussed. Let f and g be two functions. The area enclosed by the graphs of these two functions between the limits of $a \le x \le b$ is given by

$$A = \int_a^b |\, f(x) - g(x) \,|\, dx.$$

The problem of using and evalulating this formula is the point of many of the exercises.

14.2. Lecture 52

Text Reference: Chapter 5, §1

Area continued.

14.3. Lecture 53

Text Reference: Chapter 5, §2

Computing the volume of a solid by slicing.

$$V = \int_a^b A(x)\,dx$$

that is, the volume of a solid can be computed by integrating the cross-section area function, $A(x)$.

Volumes of Solids of Revolution. This is a special case of a solid whose cross sectional areas we know. Suppose we have a function f

defined over the interval $[a, b]$. Rotate the region under the graph of f around the x-axis. The volume of the resulting solid is given by

$$V = \int_a^b \pi[f(x)]^2\,dx$$

Many variations on the same theme were also discussed.

14.4. Lecture 54

Text Reference: Chapter 5, §3

Slicing continued: Volumes of solids that are described by the geometry of their cross-sections.

An alternate approach to calculating volumes was discussed: Volumes of solids of revolution by the methods of cylindrical shell.

Suppose we have a function f which is nonnegative on the interval $[a, b]$. Rotate the region below the graph of f around the y-axis. The volume of the resulting solid is given by

$$V = \int_a^b 2\pi x f(x)\, dx$$

More on monday!