⇑ toc

8. Working with the Definitions

We now come to that point in our discussion when we look at the rigorous definition of limit. Once again, let me remind you of our only description of limit: Pedestrian Description.

8.1. Motivating the Definition

The word that dominates the description of limit throughout this collection of electronic bucket of bits is the word "close" or "closer."

The Meaning of Close. Let $a \in \mathbb{R}$, and $\epsilon > 0$ a positive number. Let the word "close" to a mean any number that is with an ϵ distance of a. (For example, if $\epsilon = .001$, then a number is "close" to a is that number is within a distance of .001 of a.)

Next task, how to describe this concept mathematically. Here, we call on our algebra. Consider the inequality:

$$|x - a| < \epsilon. \tag{1}$$

What is the solution set to this inequality? Recall from your elementary algebra that the expression, $|x - a|$ has an interpretation:

$$|x - a| = \begin{array}{l}\text{the distance between}\\ x \text{ and } a.\end{array} \tag{2}$$

Thus the solution to (1) is all numbers x whose distance from a is less than ϵ. This is a direction translation into English of (1), given the interpretation of (2).

Summary of Closeness: Closeness to a is described by the inequality

$$|x - a| < \epsilon, \tag{3}$$

where $\epsilon > 0$ is a measure of the closeness to a.

Exercise 8.1. Let $a \in \mathbb{R}$ be a number. Describe what it means for x to be close to a in the sense of (3). Use the symbol $\delta > 0$ as a measure of closeness to a.

Exercise 8.2. Let $y = f(x)$ be a function, and $L \in \mathbb{R}$, a number. Describe, in the sense of (3), what it means for the values of f to be close to the number L. Use $\epsilon > 0$ as a measure of closeness

Definition of Limit. Now let's tackle the problem of describing in a very precise way the concept of limit. We want to give meaning to the symbol

$$\lim_{x \to a} f(x) = L.$$

Roughly speaking we want to describe the concept that when x is close to a, $f(x)$ is close to L; furthermore, we can make $f(x)$ as close to L as we wish by making x be sufficiently close to a.

Let's elevate this description to the stature of a boxed-in point.

> *New Pedestrian Description* (Revised)
> "$\lim_{x \to a} f(x) = L$ means as x gets closer and closer to a, $f(x)$ gets closer and closer to L; furthermore, $f(x)$ can be made arbitrarily close to L by making x sufficiently close to a.

That last phrase is new, but it conveys the meaning of limit more exactly. It describes the essence of limit better: The values of f can be made to be as close to the limiting value, L, as we wish, by putting x sufficiently close to a.

Let's try to translate the *New Pedestrian Description* into a mathematically more precise statement.

Let $\epsilon > 0$ be a measure of how close we want the values of f to be to L. Then (by EXERCISE 8.2) for the values of f to be close to L we require

$$|f(x) - L| < \epsilon. \tag{4}$$

Now let's look at our phrase to be translated: "We can make $f(x)$ as close to L as we wish (the would now be (4)) by making x sufficiently close to a." To force $f(x)$ close to L, we must make x "sufficiently close to a." I interpret this phrase as there must be some choice of closeness to a, such that if x is that close to a, $f(x)$ is close to L. Translation: There is some $\delta > 0$ (some measure of closeness to a) such that if

$$|x - a| < \delta \tag{5}$$

then

$$|f(x) - L| < \epsilon.$$

Now, let's recap: We say

$$\lim_{x \to a} f(x) = L,$$

provided, for any $\epsilon > 0$ (translation: for any measure of closeness of L we want to be), there is some $\delta > 0$ (translation: there is a measure of closeness to a), such that if $|x - a| < \delta$ (translation: such that if x is that close to a) then $|f(x) - L| < \epsilon$ (translation: $f(x)$ is the pre selected measure of closeness to L).

The next section gives a formal statement of this and illustrates with examples.

8.2. The Definition of Limit

Now lets' formally state the meaning limit.

Definition 8.1. Let $y = f(x)$ be a function, and a, $L \in \mathbb{R}$. We say that

$$L = \lim_{x \to a} f(x) \tag{6}$$

provided for any number $\epsilon > 0$, there exists a number $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and

$$0 < |x - a| < \delta$$

then,

$$|f(x) - L| < \epsilon \tag{7}$$

Definition Notes: We make a few comments in the form of bulleted paragraphs.

- The reasoning behind this definition is discussed in detail in Motivating the Definition.

- The condition $0 < |x - a| < \delta$ in (7) states that x is within a δ's distance from a, but *different* from a.

To illustrate this definition, we present a series of examples and exercises.

Example 8.1. (**Limit of a Linear Function**) Consider the function $f(x) = 2x + 5$. *Prove* that $\lim_{x \to 3} (2x + 5) = 11$.

That was long winded ... it really wasn't as difficult as it first appears. I include microminature steps in that last example. Let's do another at a more accelerated pace.

Example 8.2. Use the $\epsilon\delta$ method to prove: $\lim_{x \to 1} (6x - 1) = 5$.

Example 8.3. (**Limit of a Linear Function**) Use the $\epsilon\delta$ method to prove: $\lim_{x \to -1} (3x - 1) = -4$.

Example 8.4. (**Limit of a Linear Function**) Use the $\epsilon\delta$ method to prove: $\lim_{x \to 3/2} (7 - \frac{2}{5}x) = \frac{32}{5}$.

Exercise 8.3. Prove $\lim_{x \to 2} (5x - 2) = 8$.

Exercise 8.4. Prove $\lim_{x \to -3} (4 - 6x) = 22$.

Exercise 8.5. Consider the general problem: $\lim_{x \to a} (mx + b) = ma + b$. For a given $\epsilon > 0$, what do you think the corresponding $\delta > 0$ can be? (Review Examples 8.1, 8.2, 8.3, and 8.4.)

I grow weary of linear function, how about the limit of a degree 2 polynomial.

Example 8.5. (**Limit of a Quadratic**) Use the $\epsilon\delta$ method to prove: $\lim_{x \to 2} x^2 = 4$.

Exercise 8.6. Use the $\epsilon\delta$ method to prove: $\lim_{x \to 5} x^2 = 25$.

Example 8.6. Use the $\epsilon\delta$ method to prove: $\lim_{x \to 4} \sqrt{x} = 2$.

Try this one.

Exercise 8.7. Use the ϵ, δ method to prove: $\lim_{x \to 9} \sqrt{x} = 3$.

For Those Who Want to Know More. The rest of this section is devoted to some important details concerning the notion of limit. These details will be of interest to certain minds.

The definition of limit, Definition 8.1, is quite general, but it is slightly inaccurate. This slight weakness in the definition actually destroys the meaning of limit (some slight weakness, huh). Let me illustrate the weakness through example.

EXAMPLE 8.7. Define a function $f(x) = x$, for $0 \leq x \leq 1$. Prove $\lim_{x \to 2} f(x)$ exists, and $\lim_{x \to 2} f(x) = 10$ *and*, $\lim_{x \to 2} f(x) = -10$!

Definition 8.2. Let f be a function having domain $\operatorname{Dom}(f)$, and let $a \in \mathbb{R}$. We say that a is an *accumulation point* $\operatorname{Dom}(f)$ provided for each $\delta > 0$, there is some $x \in \operatorname{Dom}(f)$ such that $0 < |x - a| < \delta$.

Definition Notes: Basically, a is an accumulation point of $\operatorname{Dom}(f)$ is there are elements $x \in \operatorname{Dom}(f)$, in the domain of f, that are arbitrarily close to a. This was the problem we had in EXAMPLE 8.7. The number 2 was *not* an accumulation point of the domain $\operatorname{Dom}(f) = [\,0, 1\,]$;

consequently, when I put $\delta = .5$, there was no element from the domain that was with .5 of the limit point $a = 2$. This made the premise of the 'if ... then' in the definition *false*. ∎

EXERCISE 8.8. The notion of *accumulation point* is intrinsically unrelated to a function, f, or its domain, $\text{Dom}(f)$. Study Definition 8.2, and finish the following definition. "Let $A \subset \mathbb{R}$ be a subset of the real number line, and let $a \in \mathbb{R}$ be a number. We say that a is an *accumulation point* of the set A, provided ... (I pass the torch to you)

EXERCISE 8.9. Let A be a set and a a number. Define (very precisely) using the language of definitions, what it means for a *not* to be an *accumulation point* of A.

Having identified the weakness of Definition 8.1, we had better reformulate the definition correctly.

Definition 8.3. Let f be a function having domain $\mathrm{Dom}(f)$, $L \in \mathbb{R}$, and $a \in \mathbb{R}$ an accumulation point of $\mathrm{Dom}(f)$. We say

$$\lim_{x \to a} f(x) = L$$

provided, for each $\epsilon > 0$, there is some $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and

$$0 < |x - a| < \delta$$

then,

$$|f(x) - L| < \epsilon.$$

This definition of limit now fixes up Example 8.7. Now, with this new definition, $\lim_{x \to 2} f(x)$ does not exist, where here, as before, $f(x) = x$ for $x \in \mathrm{Dom}(f) = [0, 1]$. The nonexistence of this limit is now the (intuitively) correct conclusion.

In a regular calculus course, this issue of an *accumulation point* does not come up. The reason for this is that most functions we consider are functions whose domains are intervals and the limiting points we approach (limit as x approaches a) are either a *endpoint* of the interval, or an interior point of the interval (belongs to the interval, but not an end point of the interval). In each of these cases, the limiting point, a, would be an accumulation point of the domain.

In another example, we looked at the problem of calculating

$$\lim_{x\to 2} \frac{x-2}{x^2-x-2}.$$

In this case the function under consideration is

$$f(x) = \frac{x-2}{x^2-x-2}.$$

This domain of this rational function is

$$\begin{aligned}\operatorname{Dom}(f) &= \{\, x \in \mathbb{R} \mid x^2 - x - 2 \neq 0 \,\} \\ &= \{\, x \in \mathbb{R} \mid x \neq -1 \text{ and } x \neq 2 \,\} \\ &= (\, -\infty, -1 \,) \cup (\, -1, 2 \,) \cup (\, 2, \infty \,).\end{aligned}$$

Consequently, $2 \notin \mathrm{Dom}(f)$ but 2 is an accumulation point of $\mathrm{Dom}(f)$ — we can get arbitrarily close to 2 with elements of $\mathrm{Dom}(f)$. Similarly, -1 is an accumulation point of $\mathrm{Dom}(f)$. In fact, *any* real number is an accumulation point of $\mathrm{Dom}(f)$. As a result, it makes sense to pose the problem: $\lim_{x \to a} f(x)$, for any real number a.

EXERCISE 8.10. Consider the function studied elsewhere,

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x^2}.$$

Find all accumulation points of the domain of this function.

8.3. The Squeeze Theorem

A very important tool used for working with limits is the *squeeze theorem*. In this section, we state the theorem and illustrate its use.

Theorem 8.4. *Let g, f, and h be functions and a, $L \in \mathbb{R}$. Suppose there is some $\delta > 0$ such that*

$$g(x) \leq f(x) \leq h(x) \qquad |x - a| < \delta, \tag{8}$$

and,

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L. \tag{9}$$

Then

$$\boxed{\lim_{x \to a} f(x) = L.} \tag{10}$$

Proof.

Theorem Notes: There is a series of remarks in the form of bulleted paragraphs.

- This theorem is stated in a typical mathematical way: *The way it is stated, is not how it is used is practice.* In the statement, one gets the impression that we are presented with three functions having various properties. Actually, the only function you have, generally, is the function f — this is you target function. This is the function the limit of which you want to compute. The other two function, g and h, the user (that's you) *constructs*.

- Condition (8) states that when x is close to a, the graph of g is below the graph of f, and the graph of f is below the graph of h. You must construct function g and h to satisfy this condition. (Near a, f is "squeezed" between g and h.)

- You must construct the functions g and h so that their limits at a are the same value, L.

- Things aren't quite as bad as they appear above. One standard way of applying Theorem 8.4 is to create an inequality of the form

$$|f(x) - L| < F(x) \qquad |x - a| < \delta,$$

such that $\lim_{x \to a} F(x) = 0$. This is enough to prove $\lim_{x \to a} f(x) = L$. Here, $F(x)$ is some expression you have created. The condition, $|x - a| < \delta$ suggests that the inequality may be valid only for values of x "close" to a. See the example below.

EXERCISE 8.11. Using Theorem 8.4, verify the remarks of the last bulleted paragraph.

Let's now look at some examples.

EXAMPLE 8.8. Argue that $\lim_{x \to 2} x^2 = 4$ using Theorem 8.4.

8.4. Infinite Limits

Let's now formulate and illustrate the technical definition of infinite limits. The concepts and mechanical skills were demonstrated in the section entitled Infinite Limits.

Definition 8.5. Let f be a function and $a \in \mathbb{R}$ a number. We write

$$\lim_{x \to a} f(x) = +\infty$$

provided, for each positive number $M > 0$, there is a positive number $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ implies } f(x) > M.$$

Example 8.9. Prove that $\lim_{x\to 0} \dfrac{1}{x^2} = +\infty$.

Exercise 8.12. Prove $\lim_{x\to 1} \dfrac{1}{(x-1)^2} = +\infty$ using Definition 8.5.

Example 8.10. Argue that $\lim_{x\to 1} \dfrac{x}{(x-1)^2} = +\infty$.

Exercise 8.13. Argue that $\lim_{x\to -2} \dfrac{x^2}{(x+1)^2} = +\infty$.

8.5. Limits at Infinity

In this section we take up the topic introduced earlier.

Definition 8.6. Let f be a function defined on an interval of the form $(\,a, +\infty\,)$ and let L be a number. We say

$$\lim_{x\to +\infty} f(x) = L$$

provided for any $\epsilon > 0$, there is some number M such that

$$x \geq M \implies |\,f(x) - L\,| < \epsilon.$$

Solutions to Exercises

8.1. The number x is close to a provided,

$$|x - a| < \delta.$$

Exercise 8.1. ■

8.2. A value, $f(x)$, of f is close to L provided,

$$|f(x) - L| < \epsilon.$$

Exercise 8.2. ■

8.3. Follow the pattern of the previous examples, and take $\delta = \dfrac{\epsilon}{5}$.

Exercise 8.3. ■

8.4. Take $\delta = \dfrac{\epsilon}{6}$.

Exercise 8.4. ■

8.5. Let $\epsilon > 0$. Assuming $m \neq 0$, take $\delta = \dfrac{\epsilon}{|m|}$. Using this value of δ, carry out the proof of the fact

$$\lim_{x \to a} (mx + b) = ma + b.$$

Use the examples (EXAMPLE 8.2, say) to guide your presentation.

Finally, what about the case $m = 0$? For a given $\epsilon > 0$, what can we take $\delta > 0$ to be? (I warn you, the answer is trivially trivial.)

Exercise 8.5. ■

8.6. Use EXAMPLE 8.5 to guide your reasoning and presentation style.

Step: 1 Establish the relation: $x^2 - 25 = (x-5)^2 + 10(x-5)$. (Hint: Write $x^2 = [(x-5)+5]^5$.)

Step: 2 Assume $0 < |x - 5| < \delta$, for $\delta \leq 1$.

Step: 3 Show $|x^2 - 25| < \delta^2 + 10\delta \leq 11\delta$.

Step: 4 Now for $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{11}$.

Etc., etc., etc., and of course, etc.

Exercise 8.6. ■

8.7. Proceed in a series of steps.

Step: 1 Establish the relation: $|\sqrt{x} - 3| = \dfrac{|x-9|}{|\sqrt{x}+3|}$.

Step: 2 Establish, $|\sqrt{x} - 3| \leq \dfrac{|x-9|}{3}$.

Step: 3 Now for $\epsilon > 0$, choose $\delta = 3\epsilon$.

Etc., etc., so on, and so forth.

Exercise 8.7. ■

8.8. ... for each $\delta > 0$, there is some $x \in A$ such that $0 < |x - a| < \delta$. (Did you receive the torch?)

Exercise 8.8. ■

8.9. There is a number $\delta > 0$ such that there is no element $x \in A$ that satisfies the inequality $0 < |x - a| < \delta$.

This can be rephrased into a position assertion rather than a negative assertion: a is not an accumulation point of A if there is a $\delta > 0$ with the property that if $x \in A$ and $x \neq a$, then $|x - a| \geq \delta$.

For example, the number 2 is not an accumulation point of $[0, 1]$. Indeed, take $\delta > 0$ then if $x \in [0, 1]$, then $|x - 2| \geq 1 > \delta$.

Exercise 8.9. ■

8.10. The function is $f(x) = \dfrac{\sqrt{x^2+1}-1}{x^2}$, for $x \neq 0$. The domain is, as just specified,

$$\operatorname{Dom}(f) = (\,-\infty, 0\,) \cup (\,0, +\infty\,).$$

It is clear that any element of $\operatorname{Dom}(f)$ is an accumulation point of $\operatorname{Dom}(f)$. The only point of interest is $x = 0$. Is this a point an accumulation point of $\operatorname{Dom}(f)$? Of course!

Consequently, the limiting problem

$$\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{x^2},$$

is well-posed.

Exercise 8.10. ■

8.11. Suppose

$$|f(x) - L| < F(x) \qquad |x - a| < \delta,$$

where $\lim_{x \to a} F(x) = 0$. Taking the inequality above, and remove the absolute value using standard algebraic methods. Thus

$$-F(x) < f(x) - L < F(x),$$

or,

$$L - F(x) < f(x) < L + F(x). \qquad |x - a| < \delta.$$

Here, and throughout this discussion, $\delta > 0$, represents a measure of closeness to a.

Finally, define $g(x) = L - F(x)$ and $h(x) = L + F(x)$. Since we are assuming

$$\lim_{x \to a} F(x) = 0,$$

then,

$$\lim_{x \to a} g(x) = \lim_{x \to a} (L - F(x)) = L$$

and

$$\lim_{x \to a} h(x) = \lim_{x \to a} (L + F(x)) = L.$$

The conditions of Theorem 8.4 have been satisfied; therefore, we can conclude

$$\lim_{x \to a} f(x) = L.$$

That's how the theorem works — in the abstract. Exercise 8.11. ■

8.12. Let $M > 0$. We want $f(x) = \dfrac{1}{(x-1)^2} > M$. Solve this inequality for x, to get $|x - 1| > 1/\sqrt{M}$. Now take $\delta = 1/\sqrt{M}$. Finish off the proof in an organized, logical, and well-written way (use the solution to Example 8.9.

Exercise 8.12. ■

8.13. Oh, you can do it! Use the same thinking pattern as the solution to EXAMPLE 8.10.

Drop me *e-mail* if you have problems ... NOT! Exercise 8.13. ■

Solutions to Examples

8.1. First state clearly what we want to prove, Definition 8.1 is restated: In order to prove

$$L = \lim_{x \to a} f(x)$$

we must show that for any $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

Now we restate it again, specialized to our situation.

In order to prove,

$$\lim_{x \to 3} (2x + 5) = 11$$

we must show that for any $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$0 < |x - 3| < \delta \quad \text{implies} \quad |(2x + 5) - 11| < \epsilon \qquad \text{(S-1)}$$

Let me rewrite (S-1) a little: for any $\epsilon > 0$, we must find a $\delta > 0$ such that

$$0 < |x - 3| < \delta \quad \text{implies} \quad |2x - 6| < \epsilon \tag{S-2}$$

Again, another refinement of (S-2): for any $\epsilon > 0$, we need to find a $\delta > 0$ such that

$$0 < |x - 3| < \delta \quad \text{implies} \quad 2|x - 3| < \epsilon \tag{S-3}$$

Here, I have used the algebraic fact: $|ab| = |a||b|$; indeed, $|2x - 6| = |2(x - 3)| = 2|x - 3|$.

Now let's restate what we want: In order to prove

$$\lim_{x \to 3}(2x + 5) = 11$$

we must show that for any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$0 < |x - 3| < \delta \quad \text{implies} \quad 2|x - 3| < \epsilon \tag{S-4}$$

Given, now, a value of $\epsilon > 0$ (ϵ must be kept a symbolic quantity), do you see what value of δ we should choose if it is to satisfy (S-4)? What if we choose $\delta = \epsilon/2$, i.e. choose δ to be one-half the value of

ϵ. For this value of δ, is the statement in (S-4) true? Of course it is! Indeed, suppose

$$0 < |x - 3| < \delta$$

then,

$$2|x - 3| < 2\delta$$

and so,

$$2|x - 3| < \epsilon \qquad \triangleleft \text{ since } 2\delta = \epsilon.$$

This argues that the implication in (S-4) holds. Of course, the inequality $2|x - 3| < \epsilon$ is equivalent to the inequality $(2x + 5) - 11| < \epsilon$, as we argued above.

Thus we have shown that for each $\epsilon > 0$ there exists a $\delta > 0$ ($\delta = \epsilon/2$ will do), such that

$$0 < |x - 3| < \delta \quad \text{implies} \quad |(2x + 5) - 11| < \epsilon.$$

But this is exactly the definition of

$$\lim_{x \to 3} (2x + 5) = 11$$

Example 8.1. ■

8.2. Let $\epsilon > 0$. Really, we want to investigate under what conditions is it true that

$$|f(x) - L| = |(6x - 1) - 5| < \epsilon.$$

Simplify now with great speed . . .

$$|f(x) - L| = |(6x - 1) - 5| = |6x - 6| = 6|x - 1|. \tag{S-5}$$

You can see quite clearly from (S-5) that $|f(x) - L|$ can be made small (less than ϵ) by making $|x - 1|$ small. If fact, if we want $|f(x) - L| < \epsilon$, it suffices having $6|x - 1| < \epsilon$. To obtain this inequality, we must have $|x - 1| < \epsilon/6$.

Based on these ruminations, for the given $\epsilon > 0$, choose $\delta > 0$ to be $\delta = \epsilon/6$. Then if

$$0 < |x - 1| < \delta$$

then

$$0 < |x - 1| < \frac{\epsilon}{6}$$

and so,

$$6|x-1| < \epsilon.$$

Thus,

$$|f(x) - L| < \epsilon. \qquad \triangleleft \text{ from (S-5)}$$

This is what we want to prove (Definition 8.1). Example 8.2. ■

8.3. Let $\epsilon > 0$. Really, we want to investigate under what conditions is it true that

$$|f(x) - L| = |(3x - 1) - (-4)| < \epsilon.$$

Simplify the left-hand side so as to yield more information,

$$|f(x) - L| = |(3x - 1) + 4| = |3x + 3| = 3|x + 1|. \tag{S-6}$$

For the given $\epsilon > 0$, choose $\delta = \epsilon/3$. Now if,

$$0 < |x - (-1)| < \delta$$

then

$$0 < |x + 1| < \frac{\epsilon}{3}$$

and so,

$$3|x + 1| < \epsilon.$$

Thus,

$$|f(x) - L| < \epsilon. \qquad \triangleleft \text{ from (S-6)}$$

But, this is exactly what we want to prove (Definition 8.1).

Example 8.3. ■

8.4. This is the same problem of the two previous example. The numbers are not so nice is all.

Let $\epsilon > 0$. We are interested in investigating condition under which

$$|f(x) - L| = \left|(7 - \frac{2}{5}x) - \frac{32}{5}\right| < \epsilon.$$

Simplify the left-hand side so as to yield more information,

$$\begin{aligned} |f(x) - L| &= \left|(7 - \frac{2}{5}x) - \frac{32}{5}\right| \\ &= \left|\frac{3}{5} - \frac{2}{5}x\right| \\ &= \frac{2}{5}\left|\frac{3}{2} - x\right| \\ &= \frac{2}{5}\left|x - \frac{3}{2}\right| \end{aligned} \tag{S-7}$$

If you haven't figured it out by now, the goal of these manipulations is to relate $|f(x) - L|$ to the $|x - a|$ expression. This guides you simplifications.

Now if $|f(x) - L|$ is to be less than ϵ, it is apparent from (S-7), that

$$\frac{2}{5}\left|x - \frac{3}{2}\right| < \epsilon$$

or,

$$\left|x - \frac{3}{2}\right| < \frac{5}{2}\epsilon$$

The Final Argument: Let $\epsilon > 0$. Choose $\delta = \dfrac{5}{2}\epsilon$. Then,

$$0 < \left|x - \frac{3}{2}\right| < \delta$$

implies

$$0 < \left|x - \frac{3}{2}\right| < \frac{5}{2}\epsilon$$

and so,

$$\frac{2}{5}\left|x-\frac{2}{3}\right|<\epsilon.$$

Thus,

$$|f(x)-L|<\epsilon. \qquad \triangleleft \text{ from (S-7)}$$

This is exactly what we want to prove (Definition 8.1).

Example 8.4. ■

8.5. Let $\epsilon > 0$ be given. We want to make $|x^2 - 4| < \epsilon$ by making $|x - 2|$ close to zero (less than the infamous $\delta > 0$). How is $|x^2 - 4|$ related to $|x - 2|$?

To obtain a better understanding of the relationship between the size of $|x^2 - 4|$ and $|x - 2|$, we write $x^2 - 4$ in an interesting way:

$$\begin{aligned} x^2 - 4 &= [(x-2) + 2]^2 - 4 \\ &= (x-2)^2 + 4(x-2) + 4 - 4 = (x-2)^2 + 4(x-2) \end{aligned}$$

Notice now that the only time x appears is in the form of $x - 2$. This trick allows us more easily to relate $|x^2 - 4|$ to $|x - 2|$. Indeed,

$$|x^2 - 4| = |(x-2)^2 + 4(x-2)| \le |x-2|^2 + 4|x-2| \tag{S-8}$$

We are interested in closeness to 2. Let $|x - 2| < \delta$, where δ is yet to be determined. Then (S-8) becomes

$$|x^2 - 4| \le |x-2|^2 + 4|x-2| < \delta^2 + 4\delta \tag{S-9}$$

Here is a final trick. We are looking for a δ value which measures closeness to $x = 2$. Once we find a δ value that works, then *any* *smaller* δ works too! Therefore, and here's the trick, *assume* that the δ we are looking for satisfies $\delta \leq 1$. But,

$$\delta < 1 \implies \delta^2 \leq \delta. \tag{S-10}$$

Thus, (S-9) is now,

$$|x^2 - 4| < \delta^2 + 4\delta \leq \delta + 4\delta = 5\delta. \tag{S-11}$$

If we want to have $|x^2 - 4| < \epsilon$ then it suffices to choose δ such that $5\delta = \epsilon$. Thus,

$$\text{take} \qquad \delta = \frac{\epsilon}{5}$$

Suppose $0 < |x - 2| < \delta = \dfrac{\epsilon}{5}$, then, from (S-11),

$$|x^2 - 4| < 5\delta = 5\frac{\epsilon}{5} = \epsilon$$

We have shown that if

$$0 < |x - 2| < \delta$$

then,

$$|f(x) - L| = |x^2 - 4| < \epsilon.$$

This is exactly Definition 8.1.

Note: What about the condition that $\delta \leq 1$? This was needed in order to get the inequality (S-10). We should backup in our proof and define δ to be the smaller of the two numbers 1 and $\epsilon/5$. That is, define

$$\delta = \min\{\, 1, \frac{\epsilon}{5}\,\}.$$

Then, $\delta \leq 1$ as required, and $\delta \leq \epsilon/5$ as needed to get inequality (S-11).

Example 8.5. ■

8.6. First analyze where we want to go. We want to be able to make $|\sqrt{x}-2|$ close to zero by making $|x-4|$ close to zero. Let's try to relate the two quantities as was done, for example, in EXAMPLE 8.5.

$$\begin{aligned} |\sqrt{x}-2| &= |\sqrt{x}-2|\frac{|\sqrt{x}+2|}{|\sqrt{x}+2|} \\ &= \frac{|(\sqrt{x}-2)(\sqrt{x}+2)|}{|\sqrt{x}+2|} \\ &= \frac{|x-4|}{|\sqrt{x}+2|} \\ &\leq \frac{|x-4|}{2} \end{aligned} \tag{S-12}$$

The last inequality come from the fact that $\sqrt{x}+2 \geq 2$ since $\sqrt{x} \geq 0$. We have shown that

$$|\sqrt{x}-2| \leq \frac{|x-4|}{2}. \tag{S-13}$$

This is precisely the kind of inequality that we need. Now let's start the proof.

The Proof: For $\epsilon > 0$, choose $\delta = 2\epsilon$. Now if

$$0 < |x - 4| < \delta \tag{S-14}$$

then,

$$\begin{aligned}
|f(x) - L| &= |\sqrt{x} - 2| \\
&\leq \frac{|x-4|}{2} && \triangleleft \text{ from (S-13)} \\
&< \frac{\delta}{2} && \triangleleft \text{ from (S-14)} \\
&< \frac{2\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

But this is the Definiton 8.1 of limit of a function. Example 8.6. ■

8.7. Before we make the arguments, we need to take a logical sidebar. As you know, the world of mathematics is the world if ... then statements. If we make a statement 'if p, then q,' the p is referred to as the premise (or hypothesis) and the q is called the conclusion. From the point of view of logic, the *only* situation wherein the statement 'if p, then q' is considered *false* is when p is a *true* statement and q is a false. Think about this statement; does it seem reasonable?

Now the problem come in. If the only time 'if p, then q' is considered a false statement is when p is true but q is false, all other situations are true; in particular, when p is *false*, the statement 'if p, then q' is considered a *true statement*. To illustrate, here is a classic, "If there is a monkey in this room, then $1 + 1 = 3$." This is considered a *true statement* — in a perverse sort of way. (We are asserting that $1+1 = 3$, *if* is true that 'there is a monkey in the room.' This sounds very lawyeresque!)

The definition of limit is reproduced here for convenience:

$$\lim_{x \to a} f(x) = L$$

provided, for each $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Now back to the problem.

Prove: $\lim_{x \to 2} f(x) = 10$.

Proof: Let $\epsilon > 0$, choose $\delta = .5$. Now we determine if the 'if … then' statement is true: if $x \in \mathrm{Dom}(f)$ and

$$0 < |x - 2| < .5, \text{then } |f(x) - 10| < \epsilon. \tag{S-15}$$

What do you think? Is it a true statement of a false one. Keep in mind the domain of the function is (artificially) restricted to $\mathrm{Dom}(f) = [\,0, 1\,]$. If $0 < |x - 2| < .5$ that puts x in the interval $1.5 < x < 2.5$. Now here's the rub: There is no x belonging to the domain of f that satisfies the premise! *The premise is false*! (Note: the premise is p='$x \in \mathrm{Dom}(f) = [\,0, 1\,]$ and $0 < |x - 2| < .5$'.) From logical principles, we conclude that (S-15) is a *true statement*; and, in turn, we conclude

$$\lim_{x \to 2} f(x) = 10. \tag{S-16}$$

Prove: $\lim_{x \to 2} f(x) = -10$.

Proof: Same as the previous proof. Let $\epsilon > 0$ be any positive number, choose $\delta = .5$, then the following statement is *true*: if $x \in \operatorname{Dom}(f) = [\,0, 1\,]$ and

$$0 < |x - 2| < .5, \text{ then } |f(x) - (-10)| < \epsilon. \tag{S-17}$$

Again, on logical principles, we conclude that

$$\lim_{x \to 2} f(x) = -10. \tag{S-18}$$

The equations (S-16) and (S-18) stand in stark contrast to each other — very counter intuitive. These calculations seem to contradict the intuitive notion of limit. There was nothing special about the numbers 10 and -10, they played no role in the arguments. We can conclude that $\lim_{x \to 2} f(x)$ equals any number you want it to be equal to. Also, the actual definition of f (recall $f(x) = x$) played no role either. We could just have well substituted any function we wish: Let $f(x) =$

$3\sin(x^3) + tan(x)$, $0 \leq x \leq 1$, then (S-16) and (S-17) holds for this function f too!

See what I mean about the little weakness in the definition? Does it not destroy completely the intuitive meaning of limit? Can you think about what the *underlying* problem is? It's not a problem with the limit L, if you will, its not a problem with the function f, its a problem with the choice of a — the point that x is getting closer and closer to. Think about before continuing with this section.

Example 8.7. ■

8.8. If you have been reading along in this article, you might have said to yourself, "We've done this problem already!" If you have thought these thoughts, I applaud you: You have created good memory pointers. We looked at this same problem in EXAMPLE 8.5. Let's look at the solution to that problem within the context of our current theorem.

Recall portions of that solution.

$$\begin{aligned} |x^2 - 4| &= |(x+2)(x-2)| \\ &= |x+2||x-2| \end{aligned} \tag{S-19}$$

We then put a restriction on the size of δ in order to obtain a desired inequality. We are interested in x close to 2, so take $\delta < 1$, then for $|x - 2| < \delta < 1$. This, in turn, implies $1 < x < 3$. Hence, for $\delta < 1$ we have,

$$3 < x + 2 < 5 \text{and, thus,} |x + 2| < 5 \tag{S-20}$$

Substituting (S-20) back into (S-19), we get

$$|x^2 - 4| = |x + 2||x - 2| < (5)|x - 2|. \quad \triangleleft \text{ assume } \delta < 1$$

We have shown that

$$\boxed{|x^2 - 4| < 5|x - 2| \qquad |x - 2| < 1.} \tag{S-21}$$

Note that

$$\lim_{x \to 2} 5|x - 2| = 0,$$

thus, by Theorem 8.4,

$$\lim_{x \to 2} x^2 = 4.$$

Example 8.8. ■

8.9. Let $M > 0$. We want to investigate under what conditions will

$$f(x) = \frac{1}{x^2} > M.$$

We'll just do the obvious, solve this inequality for x,

$$\frac{1}{x^2} > M$$

$$x^2 < \frac{1}{M}$$

or,

$$|x| < \frac{1}{\sqrt{M}}.$$

This series of inequalities are reversible: If the last inequality is true, then that implies the first inequality is true too. Take, therefore,

$$\delta = \frac{1}{\sqrt{M}},$$

Then if

$$0 < |x| < \delta \qquad \triangleleft \text{ note } a = 0$$

implies

$$x < \frac{1}{\sqrt{M}}$$

and so implies,

$$f(x) > M,$$

as discussed above. We have shown that for any $M > 0$, there exists a $\delta > 0$ (take $\delta = 1/\sqrt{M}$) such that

$$0 < |x| < \delta \text{ implies } f(x) > M.$$

Example 8.9. ■

8.10. This one is a little harder than previous ones. Let $M > 0$, we want to investigate under what conditions

$$f(x) = \frac{x}{(x-1)^2} > M.$$

Whereas this inequality can be solved for x (try it! Please, for the practice), we take the high road instead. Put a restriction on δ: require $\delta < 1/2$. Then if $|x - 2| < \delta < 1/2$, then

$$\frac{1}{2} < x < \frac{3}{2} \qquad \delta < \frac{1}{2} \tag{S-22}$$

Then,

$$\frac{x}{(x-1)^2} > \frac{1/2}{(x-1)^2} \qquad \delta < \frac{1}{2}$$

Thus, in order to make $f(x) > M$, it suffices to make

$$\frac{1}{2(x-1)^2} > M.$$

Now this inequality can be solved rather easily:

$$|x-1| < \frac{1}{\sqrt{2M}}.$$

Given these preliminary calculations, we are now ready to prove the result. Given $M > 0$, choose $\delta = 1/\sqrt{2M}$. Then for

$$0 < |x-1| < \delta$$

we have,

$$|x-1| < \frac{1}{\sqrt{2M}}$$

Now squaring and inverting we get

$$\frac{1/2}{(x-1)^2} > M$$

But, we restricted $\delta < 1/2$ and so $1/2 > x$ as was seen above. Thus

$$f(x) = \frac{x}{(x-1)^2} > \frac{1/2}{(x-1)^2} > M.$$

We have shown that for $M > 0$, choose $\delta < 1/\sqrt{2M}$, then

$$0 < |x - 1| < \delta \text{ implies } f(x) > M.$$

This is Definition 8.5.

Notes Notes: Here is a few notes.

- Did you solve the inequality: $x/(x-1)^2 > M$? Write in the form of the degree 2 polynomial, and use the quadratic formula. You should get

$$\frac{2M + 1 - \sqrt{4M+1}}{2M} < x < \frac{2M + 1 + \sqrt{4M+1}}{2M}.$$

Question: Is this an interval containing $x = 1$? Can you show this?

- Why did I choose $\delta < 1/2$ rather than $\delta < 1$ as I have done so often in past examples? (Carefully, go over the above argument, and modify it under the assumption that $\delta < 1$ — see what goes wrong.)

■

Example 8.10. ■