



2.2. Instantaneous Velocity

Assuming that you are not familiar with the technical aspects of this section, when you think about it, your knowledge of velocity is limited. In terms of your own mathematical background, there is only one type of velocity you can deal with: *Constant Velocity*. To make matters worse, there is really only one formula for dealing with velocity: the famous formula,

$$\text{distance} = \text{velocity} \times \text{time}. \quad (4)$$

But the good news is that simple algebra is sufficient to solve many problems involving constant velocity. The bad news is that, based on life's experiences, velocity is hardly ever constant!

One way around this problem is through the notion of *average velocity*. Even though the average velocity concept is very useful in many different situations, it is inadequate for a deeper study of the dynamics of motion of a particle. For this reason, we need a better understanding of velocity. This requires us to move to a higher mathematical plane: The *Calculus* level.

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It was the *limit* concept that enabled mathematicians to move from the *algebraic level* to the *Calculus level*.

First, we begin with a ...

Review of Average Velocity. Suppose, for simplicity, a particle is moving in a straight line, and this straight line is the traditional x -axis. At time $t = t_0$ the particle is at position $x = a$, at a later time, $t = t_1$, the particle is observed to be at position $x = b$. Thus, the particle has moved from point a to point b during the time interval $[t_0, t_1]$. By definition, the *average velocity* of the particle *during the time interval* $[t_0, t_1]$ is

$$v_{\text{avg}} := \frac{b - a}{t_1 - t_0}. \quad (5)$$

Or, in mere words, the average velocity is the distance traveled ($b - a$) divided by the time ($t_1 - t_0$) needed to travel that distance.

EXERCISE 2.2. It is possible for v_{avg} to be *positive*, *negative*, or *zero*. Explain each case physically.

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What is average velocity?

v_{avg} is the constant velocity the particle would have to travel at in order to go from position $x = a$ to $x = b$ during the time interval $[t_0, t_1]$, if the particle was moving at constant velocity.

(But, of course, if this was a teenage particle its velocity would definitely *not* be constant!)

An algebraic proof of this **assertion** follows. Here, we assume that the particle is moving at a constant velocity, v_{const} , and that we have two data points: at time $t = t_0$, the particle is at $x = a$; and at time $t = t_1$, the particle is at $x = b$.

Recall equation (4), and its variant,

$$\text{distance} = (\text{velocity}) \times (\text{time}),$$

or,

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$$\text{velocity} = \frac{\text{distance}}{\text{time}}, \quad (6)$$

which is valid only in the case of constant velocity. Thus,

$$\text{distance} = b - a$$

$$\text{velocity} = v_{\text{const}}$$

$$\text{time} = t_1 - t_0.$$

Substitute into (6) to get

$$\begin{aligned} v_{\text{const}} &= \frac{b - a}{t_1 - t_0} \\ &= v_{\text{avg}}. \end{aligned} \quad (7)$$

The last equality in line (7) comes from the definition of average velocity in equation (5).

EXERCISE 2.3. (Skill Level 0) An automobile was observed to travel 30 miles in one-half hour. What is the average velocity of the automobile? What is the average velocity as measured in ft/sec?

Instantaneous Velocity. Now let's begin our discussion of the new concept of instantaneous velocity as an application to the limit concept. Suppose we have a particle moving in a straight line (the x -axis, say). Define a function as follows: at any time t ,

$$f(t) := \text{position on } x\text{-axis at time } t.$$

Problem: Fix a time point $t = t_0$, define/calculate the notion of the instantaneous velocity of the particle at time $t = t_0$; i.e., we want to know the velocity of the particle at the *instant* in time $t = t_0$. (I say 'define/calculate,' because this is a 'new' notion (define), yet, I want a calculating formula (calculate).)

This notion of instantaneous velocity, intuitively, is characterized by the position (as determined by the function f) of the particle around the time in question, $t = t_0$. Such a phrase suggests the limit concept. (see [above](#).)

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Let us begin by interpreting various quantities: Let $h > 0$ be a positive (time) value,

t_0 = the time of interest

$t_0 + h$ = the time h time units after t_0

$f(t_0)$ = position at time t_0

$f(t_0 + h)$ = position at time $t_0 + h$.

Also,

$f(t_0 + h) - f(t_0)$ = distance traveled during the
time interval $[t_0, t_0 + h]$

and finally,

$$\frac{f(t_0 + h) - f(t_0)}{h} = \begin{array}{l} \text{average velocity of the particle} \\ \text{during the interval } [t_0, t_0 + h] \end{array} \quad (8)$$

The last equation (8) follows from the definition of **average velocity**, and the observation that the length of the time interval $[t_0, t_0 + h]$ is $(t_0 + h) - t_0 = h$.

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To continue, let us take a particular function so we can make some numerical calculations to illustrate the idea. Take,

$$f(t) = t^2, \text{ and } t_0 = 2.$$

Note that in equation (8), t_0 is fixed (the time point of interest), the only variable quantity is the time increment h . We are interested in studying the velocity of the particle in smaller and smaller time intervals around the point of interest $t = t_0$. The point of the above discussion is that the expression in (8) depends on the value of h ; hence defines a *function* of h . Define, therefore,

$$v_{\text{avg}}(h) := \frac{f(t_0 + h) - f(t_0)}{h} \tag{9}$$

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Now, for the purpose of illustration, we are taking $f(t) = t^2$, and $t_0 = 2$. Then, in this case, (9) becomes

$$\begin{aligned}v_{\text{avg}}(h) &= \frac{f(t_0 + h) - f(t_0)}{h} \\&= \frac{f(2 + h) - f(2)}{h} \\&= \frac{(2 + h)^2 - 4}{h} \\&= \frac{(4 + 4h + h^2) - 4}{h} \\&= \frac{4h + h^2}{h} \\&= 4 + h.\end{aligned}$$

Thus,

$$\boxed{v_{\text{avg}}(h) = 4 + h} \tag{10}$$

Equation (10) is equation (8) specialized to $f(t) = t^2$, and $t_0 = 2$.

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Now you can clearly see the meaning of my **comment** above: In equation (10) we have the average velocity in an interval around the time point $t_0 = 2$ of interest is written explicitly as a function of h , the length of that interval.

What happens as the length of the time interval gets smaller; that is, what happens to average velocity as h gets closer and closer to 0? It should be obvious from (10): If $h \approx 0$, then $v_{\text{avg}} \approx 4$. I'll construct a table anyway:

$v_{\text{avg}} = 4 + h$							
h	1.0	0.5	0.1	0.05	0.01	0.005	0.001
v_{avg}	5.0	4.5	4.1	4.05	4.01	4.005	4.001

Let's interpret the entries of the table. I'll assume the scales of measurements are *feet* and *seconds*. For example, $v_{\text{avg}}(.01) = 4.01$ means that from time $t = 2$ to $t = 2.01$, the particle averaged 4.01 ft/sec; $v_{\text{avg}}(.001) = 4.001$ means that from time $t = 2$ to $t = 2.001$, the particle averaged 4.001 ft/sec. Finally, not included in the table is that the particle averaged 4.000001 ft/sec over the time interval $[2, 2.000001]$.

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Based on these interpretations, what are we willing to say about the velocity of the particle *at* $t = 2$? Dare we say that the particle must be moving at a velocity of 4 ft/sec at time $t_0 = 2$?

Apparently, as h gets closer and closer to 0, the corresponding value of v_{avg} appears to be getting closer and closer to 4. This is the *Pedestrian description* of limit. Therefore we write,

$$v_{\text{inst}}(2) = \lim_{h \rightarrow 0} v_{\text{avg}}(h) = 4$$

This formula does not tell the whole story, I have specialized it to this example.

Summary: Suppose a particle moves in a straight line (on the x -axis) and at time t the position of the particle is $f(t)$. Then the *instantaneous velocity* at time t is given by

$$\boxed{v(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}} \quad (11)$$

Provided such a limit exists.

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Note that I have dropped the notation t_0 and replaced it with just t —this was just a notational device used in the discussion. I have dropped the use of v_{inst} and replaced it with v , and finally, I have replaced v_{avg} with what it's equal to (see (9)).

In the previous discussion, we saw how the trend in the (average) velocity the particle over smaller and smaller intervals containing the time value of $t_0 = 2$ determined its instantaneous velocity at time $t_0 = 2$. Further, we restricted ourselves to intervals of the form $[t_0, t_0 + h]$, where $h > 0$. What about other kinds of intervals containing t_0 .

EXERCISE 2.4. Let $f(t)$ be the position on the x -axis of a particle at time t ; let t_0 be a (time) point of interest, and let $h < 0$. Establish a formula for $v_{\text{avg}}(h)$, the average velocity of the particle over the interval $[t_0 + h, t_0]$ (References: (5) and (9).)

EXERCISE 2.5. It seem rather unnatural to take t_0 as either the left-hand end point of the time interval or the right-hand end point of the interval. Perhaps, a more “natural” course is to take t_0 as the center point of a time interval around t_0 . Suppose the interval is of the form:

$[t_0 - h, t_0 + h]$, where $h > 0$. Construct a formula for $v_{\text{avg}}(h)$, the average velocity of the particle over this interval.

2.3. Tangent to a Curve

We begin by stating the basic problem of this section, which, at this introductory level to calculus, is usually thought of as the origin of differential calculus. In most calculus books, this problem is always advertised as the ...

Fundamental Problem of Differential Calculus. Given a *curve* and a *point* on that curve, define/calculate the equation of the line tangent to the given curve at the given point.

History. Mathematically, the notion of tangent line has already been defined in your past history. If you have taken a course (in high school), in *Plane Geometry*, you would have come across the following theorem due to Euclid:

Given a *circle* in the plane, and a *line* in the plane, then exactly one of the following is true: (1) the line *does not* touch the circle;

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(2) the line touches the circle at *one* point; (3) the line touches the circle at *two* points.

Remarks: In case (2), we say the line is *tangent* to the circle. The line described in case (3) is referred to as a *secant line*.

This is the sum total of your knowledge of tangent lines. Yet, it is sufficient to give you an intuitive notion of what is meant by a tangent line to an arbitrary curve.

Solution to the Problem. We need a curve. This problem will eventually be solved in greater generality, but until then we take a simple, yet important, case. Take the curve to be of the form:

$$\text{The Curve: } y = f(x) \tag{10}$$

Now we need a point on the curve.

$$\text{The Point: } P(a, f(a)) \tag{11}$$

We want to construct the equation of a line having the property that the line passes through the given point $P(a, f(a))$, and having a certain intuitive property: tangency. We need to define our line so that it has this vaporous property.

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In order to construct the equation of a line we need one of two sets of information about that line: (1) We need two points on the line; (2) we need one point on the line and the slope of the line. Note that we have neither of these two. This is the basic problem — we don't have enough information. We have one point, but we don't have a second point, nor do we have the slope of the line.

The tack we take to solve the **problem** is to define/calculate the *slope* of the imagined tangent line. If we have the slope, then we can write down the equation of the tangent line: we have a point and the slope.

To construct a line we need a second point. Let $h > 0$ be a small number. Consider $x = a$ and $x = a + h$. These two values of x are h units apart. Now look at the corresponding points on the curve:

$$P(a, f(a)) \quad Q(a + h, f(a + h)).$$

If h is small, the point Q is close to the point P . Draw a line through these two points. This line is called a *secant line*. The slope of this

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line can be computed as

$$m_{\text{sec}}(h) = \frac{f(a+h) - f(a)}{(a+h) - a}$$

Thus,

$$m_{\text{sec}}(h) = \frac{f(a+h) - f(a)}{h}. \quad (12)$$

Notice that I have represented the slope of the line through the points P and Q as a function of h ; this seems reasonable since Q is determined by the value of h and so the slope of this line depends on the choice of Q , which in turn depends on h .



Figure 1

The line through P and Q : the secant line approximates $y = f(x)$ near P .



Figure 2

An approximating secant line shown with the hypothetical tangent line.

Figure 1 depicts the secant line through the points P and Q ; its slope is given in equation (12). While in **Figure 2** the tangent line has been included. Now imagine, if you can, how the picture changes as h gets

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closer and closer to 0: the point Q moves along the curve getting closer and closer to the point P ; the secant line rotates around the pivot point P and *becomes more and more tangent-like*. Now if the secant line is looking more and more like a tangent line, then the *slope* of the secant line must be getting closer and closer to the slope of the (imagined) tangent line.

Let's summarize the major points of the above discussion: As h gets closer and closer to 0, the slope of the secant line, $m_{\text{sec}}(h)$, we imagine, will get closer and closer to m_{tan} , we imagine. But this is the *Pedestrian description* of limit!

$$\begin{aligned} m_{\text{tan}} &= \lim_{h \rightarrow 0} m_{\text{sec}}(h) \\ &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \quad \triangleleft \text{by (12)} \end{aligned}$$

or,

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

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Let's have some numerical calculations for you unbelievers in the peanut gallery.

EXAMPLE 2.2. Consider the function $f(x) = \sqrt{x}$, and take $a = 4$. Set up a table of secant slopes and try to determine the slope of the line tangent to the graph of f at $a = 4$.

The table given in the solution to **EXAMPLE 2.2**, only positive values of h were used. What about negative values of h ? Perhaps the use of negative values of h lead us to an entirely different conclusion than before.

EXERCISE 2.6. Take the table given in the solution to **EXAMPLE 2.2**, negate each of the values of h , and re-calculate the corresponding values of m_{sec} .

That's enough for an introduction to the concept of limit and how it is applied to solve the *fundamental problem of differential calculus*. You'll get plenty more in your regular calculus course as well as in these tutorials.

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Summary: The slope of the line tangent to the curve $y = f(x)$ at $x = a$ is given by

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

2.4. Rate of Change

This is another very important application to the limit concept, and is actually a generalization to **instantaneous velocity**.

There is a couple of start up concepts to be reviewed/introduced first.

The Notation of Change. If x is a number, and we change the value of x a little bit, then this new number can be represented by the notation $x + \Delta x$. The amount of change in the value of x is given

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by $(x + \Delta x) - x = \Delta x$. If $\Delta x > 0$, then the value of x has increased to $x + \Delta x$; if $\Delta x < 0$, then the value of x has decreased. Summary:

$x =$ given

$x + \Delta x = x$ is changed a little

$\Delta x =$ the amount of change in x .

In a similar way, if we have a second variable y , then we can use the same notational conventions:

$y =$ given

$y + \Delta y = y$ is changed a little

$\Delta y =$ the amount of change in y .

Change Induces Change. Suppose we have two variables x and y . And these two variables are related: $y = f(x)$. Think of x and y as fixed for now. Now, y is the value of f that corresponds to x . Suppose we change x a little bit: from x to $x + \Delta x$. Question: If x changes

by an amount of Δx , how much does the corresponding value of y change?

$$\begin{aligned}x &\longrightarrow f(x) \\x + \Delta x &\longrightarrow f(x + \Delta x)\end{aligned}$$

As the independent variable changes from x to $x + \Delta x$, the corresponding dependent variable changes from $f(x)$ to $f(x + \Delta x)$. Since these are “ y ”-values, the amount of change in the y -direction is

$$\Delta y = f(x + \Delta x) - f(x). \quad (13)$$

Notational Trick: We have denoted by y the value $f(x)$, then

$$y + \Delta y = f(x) + (f(x + \Delta x) - f(x)) = f(x + \Delta x). \quad (14)$$

Thus, we have the very pleasant representation: As “ x ” changes from x to $x + \Delta x$, “ y ” changes from y to $y + \Delta y$; or, symbolically,

$$\begin{aligned}x &\longrightarrow y \\x + \Delta x &\longrightarrow y + \Delta y.\end{aligned}$$

Here is the point of these paragraphs: A change in the x variable induces a change in the y variable. A change of Δx induces a change of Δy .

EXAMPLE 2.3. A simple example to illustrate the point. Let $f(x) = x^2$. Discuss how changes in x induce changes in the y .

Average Change Induced. In the preceding paragraphs we saw how change in one variable induced change in the other variable.

$$y = f(x)$$

$$\Delta x = \text{change in } x$$

$$\Delta y = \text{corresponding (or induced) change in } y.$$

Recall,

$$\Delta y = f(x + \Delta x) - f(x).$$

Question: What is the average change in y given the change in x ? The answer to that question is

$$\frac{\Delta y}{\Delta x} = \text{The average change in } y \text{ per } \textit{unit} \text{ change in } x. \quad (15)$$

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To understand more fully the interpretation given in equation (15), we must look at some applications. In fact, it is the applications that motivate this topic.

EXAMPLE 2.4. The radius of a balloon is measured to be r , consequently, the volume of the balloon is $V = \frac{4}{3}\pi r^3$. Additional air is introduced into the balloon which increases the radius by Δr . What change in the volume, ΔV , does the change, Δr , induce?

Instantaneous Rate of Change. We now come to the point of all this preliminary discussion. Let's illustrate the concept through example.

EXAMPLE 2.5. (EXAMPLE 2.4 continued.) The radius of a balloon is measured to be r , consequently, the volume of the balloon is $V = \frac{4}{3}\pi r^3$. Additional air is introduced into the balloon which increases the radius by Δr . What is the rate at which volume is changing with the radius 4 in.?

Concept: Instantaneous Rate of Change. Let's now abstract the previous example and define the concept of instantaneous rate of change.

The Setup. Suppose we have a function $y = f(x)$. We make the following definitions.

$$y = f(x)$$

$$\Delta x = \text{change in } x$$

$$\Delta y = \text{corresponding (or induced) change in } y.$$

Recall,

$$\Delta y = f(x + \Delta x) - f(x)$$

The *instantaneous rate of change of y with respect to x* is defined to be

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (16)$$

As a notation to reference this concept, it is customary to write

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

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Thus,

$\frac{dy}{dx}$ = The (instantaneous) rate of change of y per unit change in x .

In **EXAMPLE 2.5** we had $V = \frac{4}{3}\pi r^3$ and we argued that

$$\frac{dV}{dr} = 4\pi r^2.$$

This is a measure of the rate at which the volume, V , changes with respect to (unit change in) r .

Solutions to Exercises

2.2. Obvious.

Exercise 2.2. ■

2.3. $v_{\text{avg}} = 60 \text{ mi/hr}$.

$$\begin{aligned}v_{\text{avg}} &= 60 \times \frac{\text{mi}}{\text{hr}} \\&= 60 \times \frac{5,280 \text{ ft}}{3,600 \text{ sec}} \\&= \frac{(60)(5,280)}{3,600} \times \frac{\text{ft}}{\text{sec}} \\&= 88 \times \frac{\text{ft}}{\text{sec}}\end{aligned}$$

Exercise 2.3. ■

2.4. Since $h < 0$, $t_0 + h < t_0$. Now at time $t = t_0 + h$, the particle is located at $a = f(t_0 + h)$, and at the later time of $t = t_0$, the particle is located at $b = f(t_0)$. Now utilizing formulate (5) we get

$$\begin{aligned}v_{\text{avg}} &= \frac{f(t_0) - f(t_0 + h)}{t_0 - (t_0 + h)} \\ &= \frac{f(t_0) - f(t_0 + h)}{-h} \\ &= \frac{f(t_0 + h) - f(t_0)}{h}\end{aligned}$$

Thus,

$$v_{\text{avg}} = \frac{f(t_0 + h) - f(t_0)}{h}$$

Note that this expression is exactly the same as (9)! Consequently, in the definition of instantaneous velocity given in equation (11).

Exercise 2.4. ■

2.5. I'll leave the details to you:

$$v_{\text{avg}}(h) = \frac{f(t_0 + h) - f(t_0 - h)}{2h}.$$

If our visualization of instantaneous velocity is correct, then it should be true that velocity can also be calculated from

$$\begin{aligned}v(t_0) &= \lim_{h \rightarrow 0} v_{\text{avg}}(h) \\ &= \lim_{h \rightarrow 0} \frac{f(t_0 + h) - f(t_0 - h)}{2h}\end{aligned}$$

Are you curious? Using $f(t) = t^2$ and $t_0 = 2$.

$$\begin{aligned}v(2) &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2 - h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{(2 + h)^2 - (2 - h)^2}{2h} \\ &= \lim_{h \rightarrow 0} \frac{(4 + 4h + h^2) - (4 - 4h + h^2)}{2h}\end{aligned}$$

Solutions to Exercises (continued)

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{8h}{2h} \\ &= \lim_{h \rightarrow 0} 4 \end{aligned}$$

Now what do you suppose the last limit is? Try $v(2) = 4 \dots$ as before! Can you think of other ways of covering t_0 with different type time intervals?

Exercise 2.5. ■

2.6. As before,

$$m_{\text{sec}}(h) = \frac{\sqrt{4+h} - 2}{h}.$$

The above will be the basis for our calculations below.

$m_{\text{sec}} = (\sqrt{4+h} - 2)/h$							
h	-1.0	-0.5	-0.1	-0.05	-0.01	-0.005	-0.001
m_{sec}	.2679	.2583	.2516	.2508	.2502	.2501	.2500

What does the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $a = 4$ appear to be? Again, it appears to be 0.25. [Exercise 2.6.](#) ■

Solutions to Examples

2.2. In this case $f(x) = \sqrt{x}$, $f(4) = 2$. If $h > 0$, then $f(a + h) = f(4 + h) = \sqrt{4 + h}$. Thus, from equation (12),

$$m_{\text{sec}}(h) = \frac{\sqrt{4 + h} - 2}{h}.$$

The above will be the basis for our calculations below.

$m_{\text{sec}} = (\sqrt{4 + h} - 2)/h$							
h	1.0	0.5	0.1	0.05	0.01	0.005	0.001
m_{sec}	.2361	.2427	.2486	.2492	.2498	.2499	.2499

What does the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $a = 4$ appear to be? Maybe 0.25, or there about? [Example 2.2.](#) ■

2.3. We have $y = f(x) = x^2$. If we increment (decrement) x a little to get $x + \Delta x$, then the value of y will be changed to another value, symbolically written as $y + \Delta y$. From (14), we have

$$\begin{aligned}y &= f(x) = x^2 \\y + \Delta y &= f(x + \Delta x) = (x + \Delta x)^2 \\&= x^2 + 2x\Delta x + (\Delta x)^2.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta y &= (y + \Delta y) - y \\&= (x^2 + 2x\Delta x + (\Delta x)^2) - x^2 \\&= 2x\Delta x + (\Delta x)^2,\end{aligned}$$

and so,

$$\Delta y = 2x\Delta x + (\Delta x)^2.$$

If we change x by an amount of Δx , then y changes by an amount of $\Delta y = 2x\Delta x + (\Delta x)^2$. (Note that the amount of change in the y -variable, Δy , depends on both the value of x and the value of Δx , the amount that the x is changed. This is typical.)

In numerical terms, let's take $x = 2$. If we increase $x = 2$ by an amount of $\Delta x = .1$, then the induced change in y is $\Delta y = 2(2)(.1) + (.1)^2 = 0.41$. A change of $\Delta x = .1$ gets, in this case, magnified to a change of $\Delta y = .41$.

Note that the induced change in y (that's Δy) depends not only on the amount of change in the x (that's Δx), but also on the x -value itself. For example,

For $\Delta x = .1$	
$x = 0$	$\Delta y = .01$
$x = 1$	$\Delta y = .21$
$x = 2$	$\Delta y = .41$
$x = 3$	$\Delta y = .61$

In all cases above, if we increase the value of x by $\Delta x = .1$, the corresponding value of y also increases (since $\Delta y > 0$). We can also

get a negative Δy . In the table below, $\Delta x = -.1$, i.e. we decrement the value of x by $.1$.

For $\Delta x = -.1$	
$x = 0$	$\Delta y = 0.01$
$x = 1$	$\Delta y = -.19$
$x = 2$	$\Delta y = -.39$
$x = 3$	$\Delta y = -.59$

Thus, decrementing the value of x also decrements the value of y — except for the case $x = 0$. Draw the graph of $y = x^2$ to visualize what is going on.

Example 2.3. ■

2.4. In application work, not everything is labeled x and y . Here, the independent variable is r , and the dependent variable is V . These two variables are functionally related by $V = f(r) = \frac{4}{3}\pi r^3$.

We follow the method of solution as exhibited in [Example 2.3](#),

$$V = \frac{4}{3}\pi r^3$$

$$V + \Delta V = \frac{4}{3}\pi(r + \Delta r)^3$$

therefore,

$$\begin{aligned}\Delta V &= \frac{4}{3}\pi[(r + \Delta r)^3 - r^3] \\ &= \frac{4}{3}\pi[3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3]\end{aligned}$$

Now the average change in volume with respect to radius is

$$\begin{aligned}\frac{\Delta V}{\Delta r} &= \frac{4}{3}\pi \frac{3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3}{\Delta r} \\ &= \frac{4}{3}\pi(3r^2 + 3r\Delta r + (\Delta r)^2)\end{aligned}$$

Let's substitute some numerical values. Suppose the radius is $r = 4$ in. Then we increase the radius by an amount of $\Delta r = .5$ in. What is the corresponding change in the volume?

$$\begin{aligned}\Delta V &= \frac{4}{3}\pi[3(4)^2(.5) + 3(4)(.5)^2 + (.5)^3] \\ &\approx 113.6209 \text{ in}^3\end{aligned}$$

The average change in volume with respect to the change in the radius is

$$\begin{aligned}\frac{\Delta V}{\Delta r} &= \frac{4}{3}\pi(3(4)^2 + 3(4)(.5) + (.5)^2) \\ &\approx 227.24187 \text{ in}^3/\text{in}\end{aligned}$$

Perhaps these calculations give us a better “feel” for average change in one variable with respect to another. I changed the radius of the sphere by $\Delta r = .5$, that induced a change in the volume of $\Delta V \approx 113.6209 \text{ in}^3$.

Now, what is the interpretation of $\Delta V/\Delta r$? *It represents the amount of change in the volume per unit change in r .* To see this interpretation, we bring forth the following analogy: Suppose you are moving at a rate of 30 mi each *half-hour* (.5 hour), how many miles are you moving *per unit hour*? The answer:

$$v = \frac{30 \text{ mi}}{.5 \text{ hr}} = 60 \text{ mi/hr.}$$

In a similar fashion we can interpret $\Delta V/\Delta r$; indeed, the inclusion of the units of measurement suggest the interpretation:

$$\frac{\Delta V}{\Delta r} \approx 227.24187 \text{ in}^3/\text{in.}$$

Thus the volume is changing at a (average) rate of 227.24187 in^3 per inch; or, more exactly, when the radius was changed from $r = 4$ to $r = 4.5$ (that's a change of $\Delta r = .5$) the volume changed at a rate consistent with 227.24187 in^3 rate. Think about it!

Example 2.4. ■

2.5. Let us summarize the general results from EXAMPLE 2.4.

$$V = \frac{4}{3}\pi r^3$$

$$V + \Delta V = \frac{4}{3}\pi(r + \Delta r)^3$$

$$\Delta V = \frac{4}{3}\pi[3r^2\Delta r + 3r(\Delta r)^2 + (\Delta r)^3]$$

and,

$$\frac{\Delta V}{\Delta r} = \frac{4}{3}\pi(3r^2 + 3r\Delta r + (\Delta r)^2) \tag{S-0}$$

Now, $\Delta V/\Delta r$ represents the (average) change in volume per *unit* change in r . What happens to $\Delta V/\Delta r$ as Δr gets *closer and closer* to 0? That is, as we change the radius *less and less*, what affect does that have on the (average) change in volume per unit change in r ? Can you see in equation (S-0), that if $\Delta r \approx 0$, then $\Delta V/\Delta r \approx 4\pi r^2$?

We are interested in the behavior of $\Delta V/\Delta r$ as Δr get closer and closer to 0 — this is the **Pedestrian description** of limit. Define *instantaneous rate of change* of V with respect to (unit changes in) x as

$$\begin{aligned}\lim_{\Delta r \rightarrow 0} \frac{\Delta V}{\Delta r} &= \lim_{\Delta r \rightarrow 0} \frac{4}{3}\pi(3r^2 + 3r\Delta r + (\Delta r)^2) \\ &= \frac{4}{3}\pi 3r^2 \\ &= 4\pi r^2.\end{aligned}$$

The quantity, $4\pi r^2$ represents the *instantaneous* rate of change of volume, V , with respect to unit change in x . For example, when the radius is $r = 4$, how fast is the volume changing? The answer

$$\text{rate} = 4\pi(4)^2 = 64\pi \approx 201.0618 \text{ in}^3/\text{in}.$$

When $r = 4$, volume can change at a rate of 201.0618 cubic inches, per inch change in the radius.

Example 2.5. ■

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