toc

10. Presentation of the Theory

In this section we present some to the theory referenced within the main tutorial. We begin with a more rigorous discussion of the definition of the definite integral.

10.1. The Fundamental Theorem of Calculus

In this section we look at a series of theorems referred to as the *Fundamental Theorems of Calculus*.

Theorem 10.1. (Fundamental Theorem of Calculus, Part I.) *Let f be continuous over the interval $[a, b]$, then antiderivatives of f exist. In particular, define a new function F on the interval $[a, b]$ by,*

$$F(x) = \int_a^x f(t)\,dt,$$

then F is an antiderivative of f; this means that for any $x \in (a, b)$, $F'(x) = f(x)$.

Proof. First note that since f is continuous on $[a, b]$, then for any $x \in (a, b)$, f will be continuous on $[a, x]$. This means that

$$\int_a^x f(t)\,dt \qquad \text{exits;}$$

consequently, the function $F(x)$ is well-defined (has meaning).

Let $x \in (a, b)$. We want to prove that $F'(x)$ exists, but also we want to prove that $F'(x) = f(x)$. The argument is clever, and uses the *Intermediate Value Theorem*. Indeed, for any $h > 0$

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$

Let $\epsilon > 0$. The function f is continuous at x so there exists a number $\delta > 0$ such that if

$$0 < |t - x| < \delta \implies |f(t) - f(x)| < \epsilon.$$

Choose h so that $0 < h < \delta$, then

$$\begin{aligned}
\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\right| \\
&= \left|\frac{1}{h}\int_x^{x+h} [f(t) - f(x)]\,dx\right| \\
&\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dx \\
&< \frac{1}{h}\int_x^{x+h} \epsilon\,dx \\
&= \frac{1}{h}\epsilon h \\
&= \epsilon
\end{aligned}$$

This proves, by way of the definition of limit that,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

but, notationally, the left-hand side is $F'(x)$; thus $F'(x) = f(x)$. □

Theorem 10.2. (Fundamental Theorem of Calculus, Part II.) *Let f be integrable over the interval $[a, b]$, and suppose there is an anti-derivative F of f over the interval (a, b). Then,*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let us give ourself any partition of the interval $[a, b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

For each i, $i = 1, 2, 3, \ldots, n$, by the *Mean Value Theorem*, choose $x_i^* \in (\, x_{i-1}, x_i \,)$ so that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)(x_i - x_{i-1}),$$

the last equality due to the assumption the F is an antiderivative of f over the interval $(\, a, b \,)$.

Then

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \\ &= \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i^*)\Delta x. \end{aligned}$$

Thus,

$$F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x. \tag{1}$$

The right-hand side of (1) is a Riemann Sum. Since f is integrable over the interval $[a, b]$, we know that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = F(b) - F(a).$$

We have argued that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

and this ends the proof. □

Theorem 10.3. *Let f be an even function over the symmetric interval $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

Proof. The proof is a very nice application of the technique of substitution of variables.

Suppose a function f is an even function over the interval $[-a, a]$, then

$$
\begin{aligned}
&\int_{-a}^{a} f(x)\,dx \\
&\quad = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \triangleleft \text{ Additivity Limits} \\
&\quad = -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx && \triangleleft \begin{cases} u = -x \\ du = -dx \end{cases} \\
&\quad = -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx && \triangleleft\ f \text{ is even} \\
&\quad = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx && \triangleleft \text{ reverse limits} \\
&\quad = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \triangleleft \text{ replace } u \text{ with } x \\
&\quad = 2\int_{0}^{a} f(x)\,dx
\end{aligned}
$$

Theorem 10.4. *Let f be an odd function over the symmetric interval $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\,dx = 0. \tag{2}$$

Proof. The proof is a very nice application of the technique of substitution of variables.

Suppose a function f is an odd function over the interval $[-a, a]$, then

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx & \\
&= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \triangleleft \text{ Additivity Limits} \\
&= -\int_{a}^{0} f(-u)\,du + \int_{0}^{a} f(x)\,dx && \triangleleft \begin{cases} u = -x \\ du = -dx \end{cases} \\
&= -\int_{a}^{0} (-f(u))\,du + \int_{0}^{a} f(x)\,dx && \triangleleft\ f \text{ is even}
\end{aligned}$$

$$= \int_a^0 f(u))\,du + \int_0^a f(x)\,dx$$

$$= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx \qquad \triangleleft \text{ reverse limits}$$

$$= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx \qquad \triangleleft \text{ replace } u \text{ with } x$$

$$= 0 \quad \square$$