



## 7. The Definite Integral

The *Definite Integral* has wide ranging applications in mathematics, the physical sciences and engineering.

The theory and application of statistics, for example, depends heavily on the definite integral; through statistics, many traditionally non-mathematical disciplines have become heavily dependent on mathematical ideas. Economics, sociology, psychology, political science, geology, geography, and many others professional fields utilize calculus concepts.

Unlike the *Indefinite Integral*, which is a function, the *Definite Integral* is a numerical value. As we shall see, on first inspection, there seems to be no relation between these two mathematical objects, but as the theory unfolds, their relationship will be revealed.

The *Definite Integral*, as has been stated already, has wide-ranging application; however the problem is the diverse backgrounds of students taking *Calculus*. Some may know a lot of *physics*, while others

may have good knowledge of electrical circuits. In a *Calculus* course, no general background in the sciences is assumed, as a result, the applications that tend to be presented are of two types:

1. *Geometric Applications*: All students have a general background in geometry. Drawing curves should be your forte. Consequently, many of the applications to the *Definite Integral* seen in traditional *Calculus* courses are geometric: Calculation of *Area*, Calculation of *Volume*, Calculation of *Surface Area*, and calculation of *Arc Length*. These applications are good in the sense that they allow the student to see some useful applications, but more importantly, the students sees the process of *constructing the application*.

2. *Physical Applications*: There are some physical applications to the *Definite Integral* usually seen in a course on *Calculus*. Because of the diverse backgrounds of students, these applications tend to be easily accessed by everyone: The physical notions of *work*, *hydrostatic pressure*, *mass*, and *center of mass*.

The point I am trying to make is that there are many, many more uses of the Definite integral beyond what you will see in a standard

*Calculus* course. Do not leave *Calculus* with the false impression of the range of application of the integral.

The *Definite Integral* is different from the *Indefinite Integral* in that the former requires an elaborate *construction*. It is the *construction process* that is *key* to many applications. In subsequent sections you will see this *construction process* unfold in many ways.

### **7.1. A Little Problem with Area**

We begin by *motivating* the construction of the *Definite Integral* with a particular application — to that of the *Area* problem.

Like, what is it? Throughout your school experiences, the notion of area has been a fundamental one. *Area* is a concept whose meaning has been built up through a series of *definitions* and *deductions* through the years of your education. Area has been inculcated into you through these years until it has become second nature to you.

Area though is *limited in its application* because it is only (mathematically) defined for a limited number of geometric shapes. Did you know that?

- Here is a brief genesis of the notion of *area*. It outlines the development of the idea from the very first time you encountered it to the present.

- **The Problem**

In this section, we mean to *extend* the notion of area to more complicated regions.

**Given.** Let  $y = f(x)$  be a nonnegative function that is defined and **bounded** over the interval  $[a, b]$ .

**Problem.** **Define/Calculate** the *area* of the region  $R$  bounded above by the graph of  $f$ , bounded below by the  $x$ -axis, bounded to the left by the vertical line  $x = a$ , and bounded on the right by the vertical line  $x = b$ .

Let's elevate this problem to the status of a shadow box.

*The Fundamental Problem of Integral Calculus:*



Let  $y = f(x)$  be a nonnegative function that is defined and **bounded** over the interval  $[a, b]$ . Define Figure 1 and/or calculate the *area* of the region  $R$  bounded above by the graph of  $f$ , bounded below by the  $x$ -axis, bounded to the left by the vertical line  $x = a$ , and bounded on the right by the vertical line  $x = b$ .

The idea of solving the *Fundamental Problem* is straight forward enough; unfortunately, in order to put down in written word this idea, it is necessary to introduce a morass of notation.

- **The Idea of the Solution**

If you were asked to approximate the area of **region  $R$** , you would probably have enough understanding of the notion of area, despite the irregular shape of the boundary, to make a good approximation.

How would you do it?



One obvious way of approximating the area of an irregularly shaped region is to use paper strips. Get paper and scissors Figure 2 and cut out a series of rectangles that span the height of the region. Overlay the region using these paper strips. The heights of the rectangles need to be cut to better fit into the region. That having been done, calculate the area of each rectangle (base times height of each), then sum up all the area calculations: This will be an approximation of the area of the region.

Intuitively, the more rectangular strips you use (their widths would necessarily be getting shorter), the better approximation of the area under the graph you paper rectangular strips would yield.

This then is the basic idea behind solving the *Fundamental Problem*: Overlay the region with a larger and larger number of narrower and narrower paper strips!

**Important.** In what follows, all the terminology, notation, and the basic concepts are introduced in much detail . . . detail that you won't see elsewhere. Therefore, if you are interested in understanding the

ideas that go into the making of the *Definite Integral*, do not skip over this section. *The details of this construction are key to the application of the definite integral!* ■

### • The Technical Details

The details of the above described construction are involved but important and we present them here.

Let the interval  $[a, b]$  be given, let  $n \in \mathbb{N}$  be a **natural number**, and  $P$  be a **partition** of the interval  $[a, b]$ :

$$P = \{x_0, x_1, x_2, \dots, x_n\}.$$

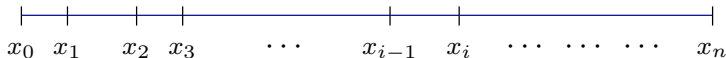
It is assumed that the labeling of the elements of  $P$  is such that

$$a = x_0 < x_1 < x_2 < x_3 < \dots < x_{n-1} < x_n = b. \quad (1)$$

These points are called *partition points*, or *nodes* of the partition.

The nodes of the partition  $P$  subdivide the interval  $[a, b]$  into  $n$  subintervals, the endpoints of which are the nodes.

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Partitioning Scheme

Let's clarify the discussion and terminology. We have divided  $[a, b]$  into  $n$  subintervals. The *partition points* given in (1) will be the end-points of these subintervals. Let's count the intervals off.

*First Sub-interval:*  $I_1 = [x_0, x_1]$ .

*Second Sub-interval:*  $I_2 = [x_1, x_2]$ .

*Third Sub-interval:*  $I_3 = [x_2, x_3]$ .

*Fourth Sub-interval:*  $I_4 = [x_3, x_4]$ .

$\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$

*The  $i^{\text{th}}$  Sub-interval:*  $I_i = [x_{i-1}, x_i]$ .

$\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$     $\vdots$

*The  $n^{\text{th}}$  Sub-interval:*  $I_n = [x_{n-1}, x_n]$ .

These subintervals are generally of different lengths. Let's establish the standard notation for the lengths of these intervals and calculate their lengths.



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$i$	<b>Interval</b>	<b>Length</b>
1	$I_1 = [x_0, x_1]$	$\Delta x_1 = x_1 - x_0$
2	$I_2 = [x_1, x_2]$	$\Delta x_2 = x_2 - x_1$
3	$I_3 = [x_2, x_3]$	$\Delta x_3 = x_3 - x_2$
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$i$	$I_i = [x_{i-1}, x_i]$	$\Delta x_i = x_i - x_{i-1}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$
$n$	$I_n = [x_{n-1}, x_n]$	$\Delta x_n = x_n - x_{n-1}$

As the notation suggests,  $\Delta x$  (with a subscript) is used to reference the length of an interval. (The value of the subscript is the ordinal number of the corresponding subinterval.) Thus,  $\Delta x_2$  is the length of the *second subinterval*, or, more generally,

$$\Delta x_i = x_i - x_{i-1}$$

is the length of the  $i^{\text{th}}$  subinterval.

**EXERCISE 7.1.** Consider the 5<sup>th</sup> subinterval in a partition. Write down the two endpoints of this interval and calculate the length of this interval. (Be sure to write the correct “ $\Delta x$ ” notation.)

**EXERCISE 7.2.** What is the sum of

$$\Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_n?$$

(Assume we have subdivided the interval  $[a, b]$  into  $n$  subintervals.)

**Summary:** This tabular approach takes a lot of time to construct and a lot of room. Let's use a different approach. Let the symbol  $i$  be used as an *index* for the interval number. If there are  $n$  intervals then we can keep track of them by the index:  $i = 1$  is the first interval;  $i = 5$  is the fifth interval. The preceding discussion and notation can then be more efficiently abbreviated by

$i^{\text{th}}$  interval:  $I_i = [x_{i-1}, x_i]$

length:  $\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, 3, \dots, n.$



Figure 3

One role these intervals play is that they subdivide the region  $R$  into  $n$  subregions; let's name these  $n$  subregions as

$$\Delta R_1, \Delta R_2, \Delta R_3, \dots, \Delta R_n.$$

Each of these subregions has area. Denote the areas of these regions by, you guessed it,

$$\Delta A_1, \Delta A_2, \Delta A_3, \dots, \Delta A_n.$$

Thus,

$$\Delta A_i = \text{the area of subregion } \Delta R_i, \quad i = 1, 2, \dots, n.$$

**EXERCISE 7.3.** What is the sum of

$$\Delta A_1 + \Delta A_2 + \Delta A_3 + \dots + \Delta A_n?$$

Another role these subintervals play in the construction process is that they form the bases of rectangles that will overlap the target area. A rectangle is characterized by its location, orientation, width and height; the area of a rectangle only depends on the latter two quantities.

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On each of the  $n$  subintervals, construct a rectangle sitting on the subinterval, and extending vertically upward. How high vertically? (That will be the height of the rectangle.) The decision about the height of the rectangle is made by choosing a point out of the subinterval; the height of the rectangle will then be the distance up to the graph of  $f$  at that point. Here are the technical details—with a morass of notation.

For each  $i$ ,  $i = 1, 2, 3, \dots, n$ , choose a point  $x_i^*$  from the  $i^{\text{th}}$  subinterval  $I_i = [x_{i-1}, x_i]$ ; i.e., choose

$$x_i^* \text{ such that } x_{i-1} \leq x_i^* \leq x_i.$$

Now, on the  $i^{\text{th}}$  subinterval,  $I_i$ , construct a rectangle sitting on the subinterval  $I_i$ , and so has base of length  $\Delta x_i$ , and extending vertically upward a height of  $f(x_i^*)$ .



It is important to note that the  $i^{\text{th}}$  rectangle overlays (almost) the  $i^{\text{th}}$  subregion  $\Delta R_i$ . The area of the  $i^{\text{th}}$  rectangle is used Figure 4 to *approximate* the area of the  $i^{\text{th}}$  subregion  $\Delta R_i$ . (Recall, we have defined  $\Delta A_i$  to be the area of  $\Delta R_i$ .)

For each  $i$ , the area of the  $i^{\text{th}}$  rectangle is

$$f(x_i^*) \Delta x_i, \quad (2)$$

for this last calculation, the formula for the area of a rectangle was used: height times base.

Since the area of the rectangle is thought of as an approximation of the area,  $\Delta A_i$ , of  $\Delta R_i$ , we have

$$\Delta A_i \approx f(x_i^*) \Delta x_i. \quad (3)$$

What we now have is a sequence of  $n$  rectangles all standing on the  $x$ -axis on their little subintervals extending vertically upwards to the graph of  $f$ . Intuitively, the total area of all the rectangles *approximates* the true area of the target region. Let's sum up all the area calculations in (3), to obtain,

$$A \approx f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3 + \cdots + f(x_n^*)\Delta x_n. \quad (4)$$

**Sigma Notation.** Notice, in (4), all the numbers we are adding up have the same form:  $f(x_i^*)\Delta x_i$ , for  $i = 1, 2, 3, \dots, n$ . In this situation,

the *sigma notation* can be used to abbreviate this sum. The notation is

$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (5)$$

The capital Greek letter  $\Sigma$  connotes summation. The notation in (5) is meant to state that we are adding up the numbers  $f(x_i^*) \Delta x_i$ , for  $i = 1, 2, 3, \dots, n$ . Additional details on the sigma notation will be given in [Section 7.4](#).

The expression on the right in (5) represents the total area of our rectangles, and is, in my mind, an approximation of the area we are trying to define/calculate.

The right-hand side of (5) has its own name. The expression

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \quad (6)$$

is called a *Riemann sum*. You'll learn more about Riemann sums later.

In our setting, the Riemann sums are approximating the area under the graph of  $f$ .

**EXERCISE 7.4.** You, theoretically, have just waded through (some) of the technical details, and, no doubt, you have read **The Idea of the Solution** concerning paper strips. Draw the parallel between the paper strip construction and the details of the technical construction. In particular, identify each of the quantities:  $n$ ,  $i$ ,  $\Delta x_i$ ,  $f(x_i^*)$ , and  $f(x_i^*)\Delta x_i$ .

- **Passing to the Limit**

Now, here's where the advanced notions of *Calculus* come in. Up to this point, it has been all notation and algebra. We must make the leap from the *approximate* to the *exact*.

What we want to do, at least conceptually, is to continue the process of approximation: partition the interval into a larger and larger number of intervals ( $n \rightarrow +\infty$ ); the length of the intervals must necessarily get smaller and smaller; overlay our approximating rectangles; add up the areas of same; and so on. We then want to look for any trends

in our rectangular approximation . . . we want to be able to discern a *limit* of these calculations. The formal details follow.

**EXERCISE 7.5.** Contrary to my statement in the last paragraph, partitioning an interval into a larger and larger number of subintervals *does not necessarily imply* that all the subintervals are “small.” Your assignment, should you decide to take it, is to partition the interval  $[0, 1]$  into 100 subintervals such that the largest one is length 0.9. (Think about how this is possible! Don’t look at the solution until you have thought it through.)

The results of **EXERCISE 7.5** suggest that we need a better understanding of the process of partitioning. In particular, we need a way of expressing the idea that we want to subdivide the target interval into a large number of small subintervals.

In fact, what we want to do is to take the limit of the **Riemann sums** as the partitions upon which the Riemann sums are computed become “finer and finer”; that is, as the number of subintervals,  $n$ , gets larger



and larger, we want the lengths of each of the subintervals in the partition to be getting smaller and smaller.

A numerical gauge is used to “control” this subdivision business. Recall that if we have a partition  $P$  of the interval  $[a, b]$ , then the usual notation is

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

For each  $i$ ,  $i = 1, 2, \dots, n$ , the length of the  $i^{\text{th}}$  subinterval is given by  $\Delta x_i = x_i - x_{i-1}$ . We now make the following definition: The *norm of the partition*  $P$  is defined to be

$$\|P\| = \max\{ \Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n \}. \quad (7)$$

The left-hand side is a symbolism for the norm of the partition  $P$ , and the right-hand side is the defining expression. In words, the norm of a partition  $P$  is the width of the widest subinterval.

The norm  $\|P\|$  is used to “control” the subdivision process. The process of subdividing the interval  $[a, b]$  into a larger and larger number of smaller and smaller subintervals can be more succinctly described

as  $\|P\| \rightarrow 0$ . Indeed, as we look at partitions  $P$  whose norm  $\|P\|$  getting smaller and smaller, the width of the widest subinterval must be going to zero. This implies that the length of *all* intervals in the partition must be going to zero, and, as the sum of their length must be  $b - a$  (see **EXERCISE 7.2**), it must be true that the number of intervals in the partition is increasing to infinity.

### • Solution to our Problem

The concept now is to take the limit of the **Riemann sums** as the *norm of the partition*,  $P$ , tends to zero. Keeping in mind equation (5), define the area under the graph of  $f$  over the interval  $[a, b]$  to be

$$A := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i. \quad (8)$$

Furthermore, because of the arbitrary way in which we chose the heights of our approximating rectangles (by choosing the  $x_i^*$  from the  $i^{\text{th}}$  subinterval  $I_i$ , and taking  $f(x_i^*)$  as the height), we require that the

limit of the right-hand side of (8) be *independent of the choice* of the points  $x_i^*$ .

As mathematicians often do, let's formalize this into a ...

**Definition 7.1.** Let  $y = f(x)$  be a nonnegative function over the interval  $[a, b]$ . The area of the region under the graph of  $f$  and above the  $x$ -axis is defined to be

$$A := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i. \quad (9)$$

provided the limit exists and does not depend on the choice of the points  $x_i^*$ .

*Definition Notes:* The limit given in Definition 7.1, equation (9), is a type of limit essentially discussed earlier. The **Riemann sum** is, roughly speaking, a function of the variable  $\|P\|$ ; consequently, in the article on **Limits**, the meaning for the limit of a function was given in **Definition 8.1**.

■ **For Those Who Want to Know More.** Actually, Riemann sums are *not* a function of the norm of the partition upon which they are based; consequently, a variation of **Definition 8.1** needs to be stated.

■ In the Section 7.5 below, **Evaluation by Partitioning**, the techniques of working with this limit are illustrated.

■ This definition is the culmination of the constructive process laboriously given above, and represents a half-solution to the **Fundamental Problem**. The problem was to define/calculate area; well, we have defined area under an arbitrary curve (with provisos), but we have no formal mechanism for the calculation of same—that comes later.

■ With all the provisos, one would wonder whether *any* function is of such a nature that the limit in **Definition 7.1** exists. I assure you that there are plenty of such functions. Indeed, any **continuous** function will due the trick. ■

**Introduction of Definite Integral Notation.** We now introduce the notation for the ultimate outcome of the constructive process. Our

basic definition is

$$A := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x,$$

We used the letter  $A$  because of the nature of the problem we were solving: An area problem. The right-hand side is the important expression. Because of its importance in modern analysis, it has its own notation to refer to it: The *Leibniz Notation*.

**Definition 7.2.** Let  $y = f(x)$  be a nonnegative function over the interval  $[a, b]$ . The area of the region under the graph of  $f$  and above the  $x$ -axis is defined to be

$$\int_a^b f(x) dx := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x, \quad (10)$$

provided this limit exists and does not depend on the choice of the points  $x_i^*$ .

*Definition Notes:* The symbolism on the left-hand side should be familiar to you. It is actually the source of the notation for the *Indefinite Integral* that we have already been using. Leibniz thought of the limit process on the left-hand side of (10) as an infinite summation process; correspondingly, he created the notation  $\int$ , which is an exaggerated version of  $S$  — for summation. The notation,  $\int_a^b$ , he thought of as a process of summation from  $a$  to  $b$ . The symbolism,  $\int_a^b f(x) dx$ , he viewed as a *transfinite* summation process of adding up areas of rectangles whose areas are  $f(x) dx$ . This view is technically incorrect, but it serves as good intuition.

- In certain simple cases, the limit in (10) can be calculated directly from the definition by using *summation rules*. (See Section 7.5 entitled **Evaluation by Partitioning** for more details.)

- **For Those Who Want to Know More.** For completeness sake, we give the reference to the precise definition of limit used above in equation (10). ■

With this notation under our belt, we can summarize:

*Solution to the Fundamental Problem:*



Let  $f$  be a nonnegative function defined over an interval  $[a, b]$ . The area under the graph of  $f$  and above the  $x$ -axis is given by

$$A = \int_a^b f(x) dx.$$

An aesthetically pleasing formula, isn't it? (*Provided* the limit exists in the sense described in [Definition 7.2](#)).

**EXAMPLE 7.1.** Consider the function  $f(x) = x^2$ ,  $0 \leq x \leq 1$ . Use the integral notation to represent the area under the graph of  $f$  and above the  $x$ -axis.

**EXAMPLE 7.2.** Symbolically represent, in terms of the definite integral, the area under the graph of  $f(x) = 3x^3 + x$  over the interval  $[1, 4]$ .

**EXERCISE 7.6.** Consider the function  $f(x) = x^4 + x$  restricted to the interval  $[-1, 5]$ . Represent symbolically, in terms of the definite integral, the area under the graph.

## 7.2. The Definite Integral

We have seen in [Section 7.1](#) the constructive process in the case of analyzing the area problem. That was just a representative application of how the constructive process is used to solve certain problems. Don't worry, I'll not go through that morass again, but we can abstract the process itself ... in summary form.

**Definition 7.3. (The Definite Integral)** Let  $y = f(x)$  be any function (not necessarily nonnegative) on the interval  $[a, b]$ .

1. *Partition*  $[a, b]$

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b,$$



into  $n$  subintervals. The  $i^{\text{th}}$  subinterval has the form  $[x_{i-1}, x_i]$ , and has length  $\Delta x_i = x_i - x_{i-1}$ , for  $i = 1, 2, 3, \dots, n$ .

2. *Choose Points*  $x_i^*$  from each interval:

$$x_{i-1} \leq x_i^* \leq x_i, \quad i = 1, 2, 3, \dots, n.$$

3. *Form Riemann Sums*:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i.$$

This sum is a Riemann sum of the function  $f$  over the interval  $[a, b]$ .

4. *Pass to the Limit*: Define integral by,

$$\int_a^b f(x) dx := \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x. \quad (11)$$

As before, the limit must exist and not depend on the choice of the arbitrarily chosen points  $x_i^*$ .

*Definition Notes:* A function  $f$  defined over an interval  $[a, b]$  is said to be *Riemann integrable* over  $[a, b]$  provided the limit in (11) exists.

■ The term **Riemann Sum** is important and worth making special note. Basically, a Riemann Sum is a sum of a function,  $f$ , being evaluated at certain intermediate points,  $x_i^*$ , times the width of the interval of partition,  $\Delta x_i$ . The limit of Riemann Sums is an integral—this is equation (11). We shall return to the important notion of Riemann sum later; one of the basic skills needed by the Calculus student, that's you, is the ability to *recognize* a Riemann Sum.

■ **For Those Who Want to Know More.** Here is the technical definition of the limit given in (11). ■

**Parsing the Notation.** The basic notation for the definite integral is

$$\int_a^b f(x) dx.$$

Here are some points concerning this notation.

*Notational Notes:* The numbers  $a$  and  $b$  are called the *limits of integration*:  $a$  is the *lower limit of integration* and  $b$  is the *upper limit of integration*.

- The limits of integration form an interval  $[a, b]$ . This interval is referred to as the *interval of integration*.

- The function  $f(x)$  is called the *integrand*.

- The symbol,  $dx$ , the differential of  $x$ , plays the same role as it did for the indefinite integral. (See the discussion of the **significance** of  $dx$ .) The symbol,  $dx$ , tells us that  $x$  is the *variable of integration*.

- The variable of integration is a *dummy variable*. The variable of integration tells us what variable the integrand is a function of; otherwise, the actual letter has no significance. For example, each of the integrals is the same as all the others:

$$\int_a^b f(x) dx = \int_a^b f(s) ds = \int_a^b f(t) dt = \int_a^b f(u) du.$$

Or, if you are having trouble thinking abstractly, all these integrals are the same:

$$\int_0^1 x^2 dx = \int_0^1 s^2 ds = \int_0^1 t^2 dt = \int_0^1 u^2 du.$$

In each instance, the integrand is the same function (it's just defined or represented using different letters). The corresponding differential notation has been changed to reflect the different dummy variable used in the definition of the integrand. **This is an important point to remember.** ■

**Purely Geometric Interpretation of the Definite Integral.** Let  $y = f(x)$  be (Riemann) **integrable** over the interval  $[a, b]$ . It is possible to give a purely geometric interpretation of

$$\int_a^b f(x) dx,$$

based on the graph of  $f$ ; this is an important and fundamental interpretation.



Here we do not assume that  $f$  is nonnegative. Imagine the graph of  $f$ . It may vary above or below the  $x$ -axis. Now imagine the rectangles sitting on the subintervals. If the graph is above the  $x$ -axis, then the rectangle extends upwards; if the graph is below the  $x$ -axis, then the rectangle extends vertically *downward*. When we form the **Riemann Sums**,

$$\sum_{i=1}^n f(x_i^*) \Delta x, \quad (12)$$

some of the values of  $f(x_i^*)$  will be positive and some *negative*. For the rectangles that are hanging downward, the  $f(x_i^*) \Delta x$  term for that rectangle will be negative: It represents the *negation* of the area of that rectangle. We then sum up these positive and negative terms (12) and pass to the **limit** to obtain the integral of  $f$ .

If you have followed this discussion, the bottom line is that the integral treats areas below the  $x$ -axis as *negative* and areas above the  $x$ -axis as *positive*. Of course, when I refer to area as *negative*, that does not

correspond to the usual notion of area. It is just a convenient device on my part to convey the idea.

To repeat, the integral treats regions below the  $x$ -axis as negative area, and regions above the  $x$ -axis as positive area, then adds all calculations together; consequently it is possible for

$$\int_a^b f(x) dx > 0 \text{ or } \int_a^b f(x) dx < 0 \text{ or } \int_a^b f(x) dx = 0$$

Perhaps I can use the term *Relative Area* or *Signed Area* to describe what the definite integral is calculating.

Consider the discussion in the previous paragraphs, then thoughtfully answer each of the following.

**EXERCISE 7.7.** Consider each of equation or inequality that follows. What is the most general statement that can be made about each?

$$(a) \int_a^b f(x) dx > 0 \quad (b) \int_a^b f(x) dx < 0 \quad (c) \int_a^b f(x) dx = 0$$

### 7.3. The Existence of the Definite Integral

Because of the complicated nature of the definition of the *Definite Integral*, one may wonder whether it is possible for *any* function to be **integrable**. In this section we briefly catalog types of functions that survive the rigors of the **constructive definition**, to emerge as integrable.

**Theorem 7.4.** (The Existence Theorem) *Let  $f$  be a function that is either **continuous** or **monotone** over the interval  $[a, b]$ , then  $f$  is integrable over  $[a, b]$ , i.e.*

$$\int_a^b f(x) dx \quad \text{exists.}$$

*Proof.* Beyond the scope of these notes.

*Theorem Notes:* This theorem justifies the earlier interest in **continuous functions**, and **monotone functions**. Identifying a function as one

whose integral exists often depends on the ability to recognize these basic function types.

- The conclusion of the **Existence Theorem** is also valid if the function  $f$  is *piecewise continuous*. A function  $f$  is *piecewise continuous* over an interval  $[a, b]$  provided  $f$  has only a finite number of discontinuities within the interval  $[a, b]$ .

- The conclusion of the **Existence Theorem** is also valid if the function  $f$  is *piecewise monotone*. A function  $f$  is *piecewise monotone* over the interval  $[a, b]$  provided we can partition the interval

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$$

such that  $f$  is monotone over each subinterval  $[x_{i-1}, x_i]$ . for  $i = 1, 2, 3, \dots, n$ .

- Virtually all functions encountered in a typical calculus course are piecewise continuous or monotone.

- A monotone function may have *infinitely many* discontinuities, yet still has a finite integral! ■



**EXAMPLE 7.3. Piecewise Continuous Function.** A piecewise continuous function can easily be constructed using piecewise definitions of functions. For example,

$$f(x) = \begin{cases} x^2 & x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

is piecewise continuous on  $[-2, 2]$ . (There is a single discontinuity within this interval at  $x = 1$ .)

**EXERCISE 7.8.** Is the function  $f$  defined in **EXAMPLE 7.3** piecewise monotone over the interval  $[-2, 2]$ ?

- **EXERCISE 7.9.** Think about different piecewise continuous functions and piecewise monotone functions (**EXAMPLE 7.3** and **EXERCISE 7.8** and all possible variations). Are these types of functions one in the same? That is, is it true that every time you write down a piecewise continuous function that function *must* be piecewise monotone? More precisely, can you give an example of a piecewise continuous function that is *not* piecewise monotone?

- **EXERCISE 7.10.** Give an example of a function that is monotone but not piecewise continuous.

(*Hint:* Sit back and try to imagine such a function.)

## 7.4. Summation Techniques

Earlier in these notes, the **sigma notation** was introduced. In this section we survey some common and useful properties of summation and introduce some famous summation formulas—all in preparation for the Section 7.5, entitled **Evaluation by Partitioning**.

The sigma notation is a way of conveying to the reader a summation process. It is important to learn to read this notation so that you can understand the thoughts that the writer is trying to communicate to you. The sigma notation is used throughout mathematics, engineering, physics, statistics, and any other fields of study that use mathematics—so pay attention!

**The Notation.** Suppose we have a collection of numbers

$$a_1, a_2, a_3, a_4, \dots, a_n \tag{13}$$

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we wish to sum. The notation for summing these numbers is

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n \quad (14)$$

Sometimes we want to sum over a *subset* of the numbers (13). Let  $k$  and  $m$  be integers such that  $1 \leq k < m \leq n$ . Now suppose we want to sum the numbers

$$a_k, a_{k+1}, a_{k+2}, \dots, a_m.$$

The notation of the sum of these numbers is

$$\sum_{i=k}^m a_i = a_k + a_{k+1} + a_{k+2} + \cdots + a_m. \quad (15)$$

*Notational Notes:* The letter  $i$  in (14) is referred to as the *index* of the sum. In (14), the index ranges from an *initial value* of  $i = 1$  to a *terminal value* of  $i = n$ .

■ The index is a *dummy variable*. It is a symbol that is used to describe the numbers to be added. The choice of the letter used to describe the summation process is not important. For example

$$\sum_{i=1}^n a_i \quad \sum_{j=1}^n a_j \quad \sum_{\alpha=1}^n a_\alpha$$

all represent the sum of the same set of numbers.

■ **Algorithm For Expanding the Sigma Notation.** There is a (simple) way of expanding the sigma notation. (By expanding the sigma notation I mean taking the left-hand side of (14), and writing it as the right-hand side of (14).) ■

The sigma notation is used in several ways: (1) As a way of summing symbols, as is the case in (14), for example; and (2) as a way of summing a particular set of numbers. We have yet to see examples of (2)—that comes now. Here are a few visual examples:

$$1. \quad \sum_{i=1}^5 i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

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$$2. \sum_{j=1}^3 \frac{j}{j+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4}.$$

$$3. \sum_{k=1}^n 2k = 2 + 4 + 6 + 8 + \cdots + 2n.$$

Notice that use of different indices in each of the examples—this is just to emphasize the point that any symbol can be used to index the sums. To fully understand these expansions, study the **Algorithm for Expanding the Sigma Notation**. Use this algorithm to expand the left-hand sides of each of the above examples. If you understand the algorithm, you will obtain the right-hand sides.

**EXERCISE 7.11.** In item (1) above, name the index, the initial value of the index and the terminal or final value of the index.

**EXERCISE 7.12.** To test your understanding of the **Algorithm for Expanding the Sigma Notation**, expand each of the following summations:

1. 
$$\sum_{i=1}^4 (i^2 - 2i)$$

3. 
$$\sum_{j=1}^5 \frac{j}{(j^2 + 2)}$$

2. 
$$\sum_{\alpha=1}^4 (5\alpha + 1)$$

4. 
$$\sum_{i=4}^7 i^2$$

**The Algebraic Properties of Summation.** When we manipulate sums of quantities we frequently take advantage of certain useful relationships the summation process enjoys.

Let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be (possibly) two sets of numbers. Let  $c \in \mathbb{R}$  be considered a constant. Then

1. 
$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

2. 
$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

**Verbalizations.** The first equation states that the sum of some numbers all of which have the same factor of  $c$  is that constant,  $c$ , times the sum of the numbers. The second equation states that the sum of a sum is the sum of the sums.

**EXERCISE 7.13.** Meditate on the phrase: “the sum of a sum is the sum of the sums.” Attempt to understand its meaning in relation to (2).

To illustrate these properties, suppose I tell you that

$$\sum_{i=1}^{20} i = 210 \text{ and } \sum_{i=1}^{20} i^2 = 2,870. \quad (16)$$

**EXERCISE 7.14.** Utilizing (16), as well as the properties of summation, (1) and (2), calculate the sum of the following:

$$1. \sum_{i=1}^{20} (i^2 - 2i) \qquad 2. \sum_{\alpha=1}^{20} (3\alpha^2 - 4\alpha)$$

- **EXERCISE 7.15.** Prove the **Algebraic Properties of Summation:** (1) and (2).

**Some Common Summation Formulas.** Here are some famous summation formula that we will be using in this section. Here are the same formulas expanded out.

$$1. \sum_{i=1}^n c = nc$$

$$3. \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$2. \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$4. \sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

**Keys to Success.** To master the use of these summation formulas, you have to understand their meaning. For example, summation formula (3) enables us to calculate the sum of the squares of the first ‘so many’ integers. Therefore, whenever you are summing up sums of squares of integers, you should think of this formula. Of course, the *first task* is to realize that you are summing up the first ‘so many’ squares of integers!



**EXAMPLE 7.4.** Here is a few sample calculations to illustrate the above formulas. Problem: Calculate each of the following:

1. 
$$\sum_{i=1}^{40} 2$$

2. 
$$\sum_{i=1}^{20} i$$

3. 
$$\sum_{i=1}^{10} i^2$$

4. 
$$\sum_{i=1}^{15} i^3.$$

(You might want to try them yourself before looking at the solution.)

In the examples above, the *index* has been the letter  $i$ . Of course, there is no significance to that particular letter. The *index* can be any letter.

**EXERCISE 7.16.** Calculate each of the following sums.

1. 
$$\sum_{i=1}^{20} 3$$

2. 
$$\sum_{j=1}^{33} j$$

3. 
$$\sum_{k=1}^{15} k^2$$

4. 
$$\sum_{\alpha=1}^{20} \alpha^3$$

**EXERCISE 7.17.** In **EXERCISE 7.16**, you just calculated sums by formulas. Do you understand what numbers you just added up? In this

exercise, write out the numbers and their sums using the following format:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \text{calculated sum.} \quad (17)$$

That is, write out the first four numbers, put “... ” then write out the last number and put the sum equal to the ... sum!

**EXERCISE 7.18.** If you were cornered by a terrible monster and he is forcing you to write the expression (17) in terms of the *sigma* notation, how would you do it to get out of the predicament?

The summation formulas can be combined with the algebraic properties of summation to solve more complex summation problems.

**EXAMPLE 7.5.** Compute the sum of each:

$$1. \sum_{i=1}^{10} i(3i + 2) \qquad 2. \sum_{j=1}^{15} (4j^3 - 9j^2 + 2) \qquad 3. \sum_{k=1}^n (3k^2 - k)$$

Now try doing some problems yourself. Work them out first before looking at the solutions. Just because you are working at home and

not turning in the problems for credit is no reason *not* to be neat. Follow my examples for presenting mathematics.

**EXERCISE 7.19.** Calculate the sum each of the following:

$$1. \sum_{i=1}^{12} (4i^2 - 6i) \quad 2. \sum_{j=1}^{15} (6j^3 - 4j^2 - 12) \quad 3. \sum_{k=1}^n (2k^2 - 3k)$$

References: The [Properties of Summation](#) and the [Summation Formulas](#).

All the formulas given [above](#) sum from 1 to  $n$ , where  $n$  is any positive integer. What if we wanted to sum over a different range of values for the index? Here's the critical but trivial observation:

$$\sum_{i=k}^n a_i = \sum_{i=1}^n a_i - \sum_{i=1}^{k-1} a_i \quad (18)$$

Read this equation and try to understand its meaning, then use it along with the algebraic properties of [summation](#) and the basic summation [formulas](#) to solve the next exercise.

**EXERCISE 7.20.** Find the sum of the following terms:

$$1. \sum_{i=20}^{40} i \qquad 2. \sum_{j=12}^{20} j^2 \qquad 3. \sum_{k=5}^{10} k^3$$

**Writing the Sigma Notation.** Given a sum, already written in the sigma notation, we have concentrated, up to this point, on expanding the notation and calculating its value using the **summation formulas**. It is also important to be able to write a series of numbers that follow some discernible pattern in terms of the sigma notation.

**Illustration 1.** For example, consider the following:

$$\frac{1}{1+1} + \frac{2}{1+2} + \frac{3}{1+3} + \cdots + \frac{n}{1+n}.$$

There is an obvious pattern here. It should be clear to you that

$$\frac{1}{1+1} + \frac{2}{1+2} + \frac{3}{1+3} + \cdots + \frac{n}{1+n} = \sum_{i=1}^n \frac{i}{1+i}.$$

■

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Here's an in-line example.

**EXAMPLE 7.6.** Write the series in sigma notation:

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n.$$

**EXERCISE 7.21.** Write the sum in sigma notation:

$$\sqrt{1+1} + \sqrt{1+4} + \sqrt{1+9} + \sqrt{1+16} + \cdots + \sqrt{1+n^2}.$$

Use good and proper notation.

In some problems in Calculus, the level of abstraction is a little higher. Try solving the one.

**EXERCISE 7.22.** Let  $f$  be a function. Write

$$(1 + f(1))^3 + (1 + f(2))^3 + (1 + f(3))^3 + \cdots + (1 + f(n))^3.$$

Let's abstract a little more.

**EXERCISE 7.23.** Let  $f$  be a function. Write

$$\sin(f(x_1)) + \sin(f(x_2)) + \sin(f(x_3)) + \cdots + \sin(f(x_k)),$$

where  $x_1, x_2, x_3, \dots, x_n$  are a pre-selected set of numbers.

The last exercise is a kind of sum that appears in the constructive definition of the *Definite Integral*. Recall the form of a **Riemann Sum**:

$$\begin{aligned} \sum_{i=1}^n f(x_i^*) \Delta x_i &= f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 \\ &\quad + f(x_3^*) \Delta x_3 + \cdots + f(x_n^*) \Delta x_n. \end{aligned}$$

Reading from left to right, the left side can be expanded using the **algorithm** to obtain the right-hand side. Now if you start from the right-hand side, you should be able to see the pattern and should be able to construct the sigma notation that is the left-hand side.

## 7.5. Evaluation by Partitioning

Certain integrals can be evaluated directly from the **definition**. Calculating an integral from its definition brings alive this constructive definition and helps us to better understand the definite integral.

In applied fields, the definite integral is estimated using numerical techniques. All of these techniques are founded on the constructive definition. Even though we will learn a very slick method of evaluating a definite integral in **SECTION 8**, this method cannot be applied to every definite integral. Some integrals *have to be estimated* using numerical techniques. Knowledge of the constructive definition of the definite integral is essential to better understanding numerical methods that are so important in the applied disciplines.

That having been said, let's pause a little while to compute a few integrals.

*General Problem:* Compute  $\int_a^b f(x) dx$  from the definition.

The first step in this process is to subdivide the interval of integration,  $[a, b]$ , into subintervals. Very often, it is convenient to subdivide the interval  $[a, b]$  into subintervals of *equal length*.

**Definition 7.5.** A *regular partition* of  $[a, b]$  is a **partition**,

$$P = \{ x_0, x_1, x_2, \dots, x_n \}$$

such that,

$$\Delta x_1 = \Delta x_2 = \Delta x_3 = \dots = \Delta x_n,$$

where  $\Delta x_i = x_i - x_{i-1}$  is the width of the  $i^{\text{th}}$  subinterval,  $i = 1, 2, 3, \dots, n$ . In this case, the partition,  $P$ , has subdivided the interval  $[a, b]$  into  $n$  subintervals all of the same length.

**Important Point.** If we want to partition the interval  $[a, b]$  into  $n$  subintervals of equal length (a regular partition) then the common width of these subintervals will be  $(b - a)/n$ ; that is

$$\Delta x_1 = \Delta x_2 = \Delta x_3 = \dots = \Delta x_n = \frac{b - a}{n}. \quad (19)$$



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As a shorthand notation, write  $\Delta x = (b - a)/n$  (without a subscript) to represent this common length.

Note also that the norm of a regular partition  $P$  is  $\|P\| = (b - a)/n = \Delta x$ .

*Calculation of  $\Delta x$  for a Regular Partition:*

If we want to subdivide the interval  $[a, b]$  into  $n$  subintervals all having the same length, the common length of these subintervals is given by

$$\Delta x = \frac{b - a}{n}$$

Take some time out to make sure you believe equation (19).

**EXERCISE 7.24.** Verify, in your own mind, equation (19).

**EXERCISE 7.25.** Suppose we have the interval  $[1, 4]$  and we want to partition it into  $n = 6$  subintervals of equal length. What is the value of their common length?

**Steps for Calculating  $\int_a^b f(x) dx$  from the Definition.**

The calculation of an integral from its definitional roots generally takes five steps:

1. Calculate the value of  $\Delta x = (b - a)/n$  (the  $n$  is kept symbolic);
2. calculate the partition points (nodes);
3. choose the intermediate points  $x_i^*$ ;
4. given that choice, calculate the exact sum of the corresponding **Riemann Sum**; and
5. having calculated the Riemann Sums; calculate the limit of the Riemann sums as the **norm** of the partition tends to zero. (See **constructive definition** of the definite integral.

**Some Notes on the Calculation of the Partition Points.** When using a **regular partition**, it is actually very easy to compute the nodes. Let me explain through a series of exercises that you should solve. Be

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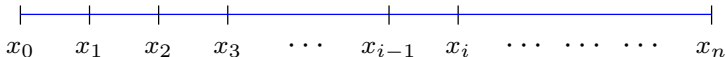
sure to read the solutions to these exercise because they contain some standard techniques for computing the nodes of a regular partition.

Try to figure this one out on your own ... it is important, not very difficult.

**EXERCISE 7.26.** The endpoints of the interval,  $a$  and  $b$ , are known. Generally, we decide to subdivide the interval into  $n$  equal subinterval, so  $n$  is known, what is unknown is the values of the partition points or nodes:

$$x_0, x_1, x_2, x_3, x_4, \dots, x_n.$$

Here's a visualization for your viewing pleasure:



Partitioning Scheme

Determine how to calculate these points using the known information:  $a$ ,  $b$ , and  $n$ .

**EXERCISE 7.27.** We want to subdivide the interval  $[1, 4]$  into 6 subinterval all of the same length. Calculate the partition points or nodes of this partition. *(Hint: Consult EXERCISE 7.26)*

Here is an (optional) exercise of the same type.

**EXERCISE 7.28.** Subdivide the interval  $[-1, 5]$  into 8 equal subintervals. Calculate the nodes of this partition. Use good notation.

**EXERCISE 7.29.** A friend of mine has partitioned an interval into 20 equal subintervals and has obtained a formula for each partition point:

$$x_i = 4 + \frac{4i}{5} \quad i = 0, 1, 2, 3, \dots, 20.$$

The problem is that he won't tell me which interval he subdivided. Can you help me out?

**Partition Limits and Regular Partitions.** The kind of limit that appears in the definition of the definite integral is a fairly complicated

one:

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i. \quad (20)$$

Let  $f$  be integrable over the interval  $[a, b]$ , this means that the limit in equation (20) exists and does not depend on the choice of the intermediate points  $x_i^*$ . Focus in on the  $\|P\| \rightarrow 0$  in (20). When  $P$  is a **regular partition**,  $\|P\| = (b - a)/n$ . As  $\|P\|$  gets closer and closer to zero, it must be true that  $n$ , the number of subintervals into which we have subdivided  $[a, b]$ , must be getting larger and larger. Observe, therefore, that for regular partitions:

$$\|P\| \rightarrow 0 \text{ is equivalent to } n \rightarrow \infty.$$

Consequently, when working with regular partitions, equation (20) is rewritten as

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x. \quad (21)$$

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Notice not only the replacement of  $\|P\| \rightarrow 0$  with  $n \rightarrow \infty$  in (21), but also the replacement of  $\Delta x_i$  with just  $\Delta x$ . Both of these replacements tend to simplify the (complex) task of calculating a definite integral from its definition.

- **For Those Who Want to Know More.** Here is the technical definition of the limit given in (21).

Let's elevate equation (21) to the status of a shadow box for easy reference later. The notation within the box is the standard notation of the **constructive definition**.

*Regular Partitions: Calculating a Definite Integral:*

Let  $f$  be **integrable** over the interval  $[a, b]$ . Then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

**Calculation of Definite Integrals.** Let's begin evaluating definite integral by checking for internal consistency. Consider the function  $f(x) = 3$ ,  $0 \leq x \leq 2$ . Graph this function up over this interval. The region under the graph is a rectangle. The base,  $b$ , of the rectangle is the length of the interval  $[0, 2]$ , which is  $b = 2$ . The height,  $h$ , of the rectangle is  $h = 3$ . Therefore, the area under the graph of  $f$  graphed over the interval  $[0, 2]$  is  $A = bh = (2)(3) = 6$ . Will the definite integral yield this same value? Let's see.

**EXAMPLE 7.7.** Calculate  $\int_0^2 f(x) dx$ , where  $f(x) = 3$ , using the constructive definition of the definite integral.

**EXERCISE 7.30.** Compute  $\int_1^4 f(x) dx$ , where  $f(x) = 5$  using the constructive definition of the definite integral.

(*Hint:* Follow the steps of **EXAMPLE 7.7**)

Here is the same problem but abstracted. The end result of the exercise is to assure ourselves that the integral *always* gives us back the area of rectangle.

- **EXERCISE 7.31.** Let  $b$  and  $h$  be positive numbers. Define a function by  $f(x) = h$ . Show, from **basic principles**, that

$$\int_0^b h \, dx = bh.$$

(*Hint:* Follow **EXAMPLE 7.7.**)

Let's move on to something a little more challenging: The area of a triangle. We still want to check whether the definite integral produces the answer we would expect it to produce. If it does, it will give us more and more confidence that the constructive definition is correct; that is, it defines what we think it defines.

**EXAMPLE 7.8.** Consider the function  $f(x) = 4x$  over the interval  $[0, 5]$ . The region under the graph of  $f$  and above the  $x$ -axis is a



triangle with base 5 and height 20 ( $f(5) = 20$ ). The area under the graph can be expressed by

$$A = \int_0^5 4x \, dx.$$

Calculate the value of this integral from basic principles.

We have seen now several examples of how to calculate the value of a definite integral. A natural question arises: In **EXAMPLE 7.8**, we obtained a value of 50, how do we know that this is really the value of the integral?

In that example, we chose as the intermediate values,  $x_i^*$ , the right-hand endpoints. The definition requires that the limit of the Riemann sums be the same value no matter what choice of the intermediate points. How do we know that if we had chosen the left-hand endpoints or the midpoints of each interval, or some randomly chosen points taken from each interval, we would always come out with a limit of 50?

- **EXERCISE 7.32.** Read the discussion of the previous paragraphs and determine the reason why no matter how we would have chosen, the limit of the corresponding Riemann Sums would have still come out to be 50.

**EXERCISE 7.33.** In **EXAMPLE 7.8** we calculated the value of

$$\int_0^5 4x \, dx = 50.$$

Review the solution to that example, and based on your review determine the value of the integral:

$$\int_0^5 -4x \, dx.$$

**EXERCISE 7.34.** Use the constructive definition of the definite integral to calculate the integral

$$\int_0^2 3x + 1 \, dx.$$

Take the intermediate points to be the right-hand endpoint of each subinterval. Before you get started, can you predict the answer based on geometric interpretation?

Let's try another where the lower limit of integration is nonzero. This exercise is done the same way as the previous; however, there is considerably more algebra—good thing you are a *master of algebra*!

- **EXERCISE 7.35.** Calculate  $\int_{-1}^2 3x^2 - 2x \, dx$  using the constructive definition of the definite integral.

## 7.6. Properties of the Definite Integral

In this section we take a survey of the general properties of the *Definite Integral*. These properties are very useful in the calculation of the integral, as well as completing your understanding of the basic integral.

All the following properties can be readily deduced from the **constructive process**. The first few properties will be quite familiar to you, as they are shared properties of the *Indefinite Integral*.

**Homogeneous Property.** Suppose the  $f$  is integrable over the interval  $[a, b]$  and  $c \in \mathbb{R}$ , then  $cf$  is integrable over the interval  $[a, b]$ , and

$$\int_a^b cf(x) dx = c \int_a^b f(x) dx.$$

**The Additive of Integrand.** Suppose the functions  $f$  and  $g$  are integrable over the interval  $[a, b]$ , then  $f + g$ , is integrable over the interval  $[a, b]$ , and

$$\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

Of course, a similar statement is true for the difference of two functions.

**Additivity of Limits.** Suppose  $f$  is integrable over the interval  $[a, b]$ , and let  $c \in [a, b]$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

**Nonnegativity of the Integral.** Suppose the function  $f$  is integrable over  $[a, b]$  and that  $f(x) \geq 0$  over  $[a, b]$ , then

$$\int_a^b f(x) dx \geq 0.$$

**Monotone Property of the Integral.** Suppose  $f$  and  $g$  are integrable over the interval  $[a, b]$  and that  $f(x) \leq g(x)$  for all  $x \in [a, b]$ , then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

**Absolute Integrability.** Suppose  $f$  is integrable over the interval  $[a, b]$ , then  $|f|$  is integrable over the interval  $[a, b]$  too, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

**Integral over Interval of Zero Length.** We occasionally come across the problem in integrating over an interval of zero length. We make the following **definition**: For any number  $a \in \text{Dom}(f)$ ,

$$\int_a^a f(x) dx = 0.$$

**Reverse Order of Integration.** When we perform **substitution**, the limits of integration sometimes get mixed around. In order to make valid the substitution method with definite integrals, we make the

following **definition**: Suppose  $f$  is **integrable** over the interval  $[a, b]$ , then

$$\int_b^a f(x) dx = - \int_a^b f(x) dx.$$

## 8. Evaluation of the Definite Integral

Despite the very complicated nature of the construction of the definite integral, there is an elementary technique for its evaluation. That is the topic of this section.

### 8.1. The Fundamental Theorem of Calculus

There is a surprising relationship between the divine simplicity of antidifferentiation and the infernal construction of the definite integral; a relationship that is set forth in the two theorems.

**Theorem 8.1.** (Fundamental Theorem of Calculus, Part I.) *Let  $f$  be continuous over the interval  $[a, b]$ , then antiderivatives of  $f$  exist. In particular, define a new function  $F$  on the interval  $[a, b]$  by,*

$$F(x) = \int_a^x f(t) dt,$$

*then  $F$  is an antiderivative of  $f$ ; this means that for any  $x \in (a, b)$ ,  $F'(x) = f(x)$ .*

**Proof.**

**Theorem 8.2.** (Fundamental Theorem of Calculus, Part II.) *Let  $f$  be integrable over the interval  $[a, b]$ , and suppose there is an antiderivative  $F$  of  $f$  over the interval  $(a, b)$ . Then,*

$$\blacksquare \quad \int_a^b f(x) dx = F(b) - F(a).$$

**Proof.**



## 8.2. The Mechanics: Simple Demonstrations

The *Fundamental Theorem of Calculus* is our big weapon for evaluating definite integrals:

$$\int_a^b f(x) dx = F(b) - F(a), \quad (1)$$

where  $F$  is *any* antiderivative of  $f$ .

**EXERCISE 8.1.** We have learned that the most general antiderivative of a function  $f(x)$  has the form  $F(x) + C$ . In the above statement, we emphasize the word ‘*any*’ antiderivative. Show that the inclusion of the constant  $C$  has no effect on the value of the integral in equation (1).

### Notational Conventions.

The right-hand side of the above equation is often abbreviated to

$$F(x)|_a^b = F(b) - F(a) \quad (2)$$

thus,

## Section 8: Evaluation of the Definite Integral

$$\int_a^b f(x) dx = F(x)|_a^b.$$

The function  $F$  is an antiderivative of  $f$ . The techniques for constructing antiderivatives was discussed exhaustively earlier. Recall, an antiderivative of  $f$  is the same as the *indefinite integral* of  $f$ :

$$\int f(x) dx,$$

and so, equations (2) becomes

$$\int_a^b f(x) dx = \int f(x) dx \Big|_a^b.$$

Let's look at a series of examples.

**EXAMPLE 8.1.** Evaluate  $\int_2^3 x^2 dx$ .

## Section 8: Evaluation of the Definite Integral

**EXAMPLE 8.2.** Evaluate  $\int_0^4 x\sqrt{x} dx$ , and interpret its value.

**EXAMPLE 8.3.** Evaluate  $\int_{-2}^{-1} x^4 - 2x dx$ .

*Summary:* Definite Integration problem,

$$\int_a^b f(x) dx$$

consists of two distinct steps:

1. **Evaluate an Indefinite Integral:**

$$\int f(x) dx$$

This step was discussed extensively **earlier**. The evaluation of an *indefinite integral* is a *function* whose derivative is  $f(x)$ .

## 2. Evaluate Indefinite Integral between the Limits:

$$\int f(x) dx \Big|_a^b$$

This step is merely numerical, and, as such, is conceptually rather simple; however, this step must be given your fullest attention as there are plenty of opportunities for error.

The first step can be rather difficult; the second, given the first step has been completed, is trivial. (Yet, error can occur, be careful). Therefore, the emphasis in *Calculus I* and *Calculus II* is on developing the *techniques* to successfully accomplish step (1).

## 9. Techniques of Evaluating Definite Integrals

In this section we take a survey of standard methods of evaluating the definite integral. This survey is by no means complete, there are many other techniques you will encounter in later calculus courses.

### 9.1. EBLO Tricks

One of the problems students have is to make a numerical calculation more difficult than it really has to be. When we evaluate a definite integral, we are calculating a number. The method we use primarily to do this is the **Fundamental Theorem of Calculus**:

$$\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a)$$

The arithmetic process on the right-hand side has several properties you can occasionally exploit. Let us refer to

$$F(x)|_a^b = F(b) - F(a) \tag{1}$$

as **EBLO**: *E*valuation *b*etween the *l*imits *O*peration.

**Homogeneity of EBLO.** This property is based on the observation that if  $c$  is a constant then

$$cF(b) - cF(a) = c(F(b) - F(a))$$

This equation can be translated using the evaluation notation as

$$\boxed{(cF(x))\Big|_a^b = c(F(x))\Big|_a^b} \quad (2)$$

Roughly speaking, when we evaluate an antiderivative between limits, and the antiderivative has a constant factor embedded in it, then this constant factor can be factored out of the evaluation.

**Verbalization.** The evaluation of a constant times a function is that constant times the evaluation of that function.

Let's illustrate the utility of this property by way of a simple example.

**EXAMPLE 9.1.** (Illustration of **Homogeneity Property**)

Evaluate  $\int_2^3 x^3 dx$ .

**Additivity Property of EBLO.** This property is derived from the simple observation that if  $F$  and  $G$  are functions then,

$$(F(b) + G(b)) - (F(a) + G(a)) = (F(b) - F(a)) + (G(b) - G(a))$$

This is translated in terms of the evaluational notation as

$$\boxed{(F(x) + G(x))\Big|_a^b = F(x)\Big|_a^b + G(x)\Big|_a^b.} \quad (3)$$

This can be verbalized by saying that the evaluation of the sum of two antiderivatives is the sum of the evaluations of each antiderivative.

**Verbalization.** The evaluation of the sum of two functions is the sum of the evaluations of each function.

The **Additive Property** of the evaluation operation can be used from left-to-right or from right-to-left.

**EXAMPLE 9.2.** (Illustration of **Additive Property**)

Evaluate  $\int_{-1}^1 x^3 - x^2 dx$ .

## Section 9: Techniques of Evaluating Definite Integrals

**EXERCISE 9.1.** Evaluate the integral  $\int_{-2}^2 x^2 - 5x^5 dx$  using the properties of **homogeneity** and **additivity** of **EBLO**.

If you were one of the unfortunates who did not properly execute the last exercise to maximum efficiency, try another, please.

**EXERCISE 9.2.** Evaluate  $\int_2^3 3x^3 + 2x^2 dx$ .

**Reversing the Order of Evaluation.** Observe the following simple algebraic property:

$$F(b) - F(a) = -(F(a) - F(b)).$$

This can be translated into the **EBLO** notation as follows:

$$\boxed{F(x)|_a^b = -F(x)|_b^a.} \quad (4)$$

**Verbalization.** The evaluation between two limits is negative the evaluation of the limits in reverse order.



Here is a simple example to illustrate the minor use of this property.

**EXAMPLE 9.3.** Evaluate  $\int_1^2 x^{-3} dx$ .

These are just some useful tips that you can use when trying to evaluate definite integrals efficiently and without error.

## 9.2. Definite Integration and Substitution

When you are working a definite integral and the technique of **substitution** is called for, how do we proceed? There is actually two ways: (1) Treat the integral as an *indefinite integral*, perform the substitution, obtain the antiderivative, then evaluate the antiderivative between the limits; or, (2) continue with the definite integral, make the substitution, and *change the limits of integral*. **Changing the limits** is explained below.

First up is an example wherein the technique of substitution is used, the solution indicates a train of thought you can use.

## Section 9: Techniques of Evaluating Definite Integrals

**EXAMPLE 9.4.** Evaluate  $\int_0^1 x\sqrt{x^2 + 3} dx$ .

**EXERCISE 9.3.** Evaluate  $\int_{-2}^0 \frac{x}{(x^2 + 1)^3} dx$ .

The next example illustrates a style you can adapt. In this approach, we combine the indefinite integration step with the definite integral, we set up the substitution, but don't substitute. Check it out.

**EXAMPLE 9.5.** Evaluate  $\int_0^{\pi/4} \sin(2x) dx$ .

An alternate approach to finding the indefinite integral as a separate step, and evaluating it between the specified limits, we can simply change the limits of integration.

**Changing the Limits of Integration.**

## Section 9: Techniques of Evaluating Definite Integrals

*Justification:* Let  $F$  be an antiderivative of  $f$ . This means

$$\int f(u) du = F(u) + C. \quad (5)$$

Let  $g$  be a differentiable function compatible for composition with  $f$ , then, by the Chain Rule,  $F(g(x))$  is an antiderivative of  $f(g(x))g'(x)$ , since the derivative of the former is the latter. This means

$$\int f(g(x))g'(x) dx = F(g(x)) + C. \quad (6)$$

As we have already seen, the technique of substitution is the equation you get when you equate (5) and (6) with the proviso that  $u = g(x)$ :

$$\int f(g(x))g'(x) dx = \int f(u) du,$$

where  $u = g(x)$ .

Now if its a *definite integral* we want to evaluate, then from (5),

$$\int_a^b f(g(x))g'(x) dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)). \quad (7)$$

## Section 9: Techniques of Evaluating Definite Integrals

The trick now is to notice that if we were to take the integral in (6) and integrate it between certain special limits:

$$\int_{g(a)}^{g(b)} f(u) du = F(u) \Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)). \quad (8)$$

The right-hand sides of (7) and (8) are identical; hence, the left-hand sides are the same,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

This is the substitution formula with limits.

*Substitution with Limits:*

Let  $f$  and  $g$  be functions. Let  $u = g(x)$  and  $du = g'(x)$ , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

This formula is not as difficult as it first seems. The process of setting up the limits of integration for the new integral is quite natural. Let's go to the examples.

**EXAMPLE 9.6.** Evaluate  $\int_0^1 x\sqrt{x^2 + 3} dx$ .

**EXAMPLE 9.7.** Evaluate  $\int_{-2}^3 (1 - 4x)^{-5} dx$ .

### 9.3. Taking Advantage of Symmetry

There are occasions in which you can take advantage of any symmetry properties that the integrand may have *over the interval of integration*. In this section we examine two situations of interest: Integrating an *even* function over an interval of the form  $[-a, a]$ ; Integrating an *odd* function over an interval of the form  $[-a, a]$ .

**Integrating an Even function.** A function  $f$  defined on a (symmetrical) interval  $[-a, a]$  is called an *even* function if it is true that

$$f(-x) = f(x) \quad \text{for all } x \in [-a, a]. \quad (9)$$

In the jargon of *analytic geometry*, we say that  $f$  is symmetrical with respect to the  $y$ -axis.

If  $f$  is an even function over the interval  $[-a, a]$ , then graphically, this means that the graph of  $f$  over the interval  $[-a, 0]$  is identical to the graph of  $f$  over the interval  $[0, a]$ . Intuitively, this would seem to imply that

$$\int_{-a}^0 f(x) dx = \int_0^a f(x) dx.$$

This intuitive result does take a bit of effort to argue analytically, but its proof does represent a nice example of the use of **substitution**. The interested student is invited, therefore, to read the proof of the following magnification of the obvious.

**Theorem 9.1.** *Let  $f$  be an **even** function over the symmetric interval  $[-a, a]$ , then*

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx. \quad (10)$$

**Proof.**

*Theorem Notes:* The utility of (10) is that it tends to simplify the numerical calculations after you have solve the *indefinite integral*. Taking advantage of symmetry this way can be very useful. ■

**EXERCISE 9.4.** Using **Theorem 9.1**, in particular, equation (10), show that

$$\int_{-a}^0 f(x) dx = \int_0^a f(x) dx$$

## Section 9: Techniques of Evaluating Definite Integrals

**EXAMPLE 9.8.** Evaluate  $\int_{-2}^2 x^2 dx$ .

**Keys to Success.** To use this technique you need to (1) have the ability to identify even functions, these are functions that are symmetric with respect to the  $y$ -axis (assuming  $y$  is the dependent variable); and (2) have the awareness of the type of interval you are integrating over: symmetric or not.

**EXERCISE 9.5.** Evaluate  $\int_{-1}^1 x^4 - 2x^2 - 1 dx$ .

**EXERCISE 9.6.** Evaluate  $\int_{-\pi/4}^{\pi/4} \cos(x) dx$ .

**EXERCISE 9.7.** Evaluate  $\int_{-1}^1 x^6 - x^3 dx$ .



**Integrating an Odd Function.** A function  $f$  defined on a (symmetrical) interval  $[-a, a]$  is called an *odd* function if it is true that

$$f(-x) = -f(x) \quad \text{for all } x \in [-a, a]. \quad (11)$$

In the jargon of *analytic geometry*, we say that  $f$  is symmetrical with respect to the origin.

If  $f$  is an odd function over the interval  $[-a, a]$ , then graphically, this means that the graph of  $f$  over the interval  $[-a, 0]$  is identical to the graph of  $f$  over the interval  $[0, a]$ , except that one graph is the reflection of the other with respect to the  $x$ -axis. Intuitively, this would seem to imply that

$$\int_{-a}^0 f(x) dx = - \int_0^a f(x) dx.$$

This intuitive result does take a bit of effort to argue analytically, but its proof does represent a nice example of the use of **substitution**. The interested student is invited, therefore, to read the proof of the following magnification of the obvious.

**Theorem 9.2.** Let  $f$  be an *odd* function over the symmetric interval  $[-a, a]$ , then

$$\int_{-a}^a f(x) dx = 0. \quad (12)$$

**Proof.**

**EXERCISE 9.8.** Using **Theorem 9.2**, in particular, equation (12), show that

$$\int_{-a}^0 f(x) dx = - \int_0^a f(x) dx$$

**EXAMPLE 9.9.** Evaluate  $\int_{-1}^1 x^3 dx$ .

**Keys to Success.** To use this technique you need to (1) have the ability to identify odd functions, these are functions that are symmetric with respect to the origin; and (2) have the awareness of the type of interval you are integrating over symmetric or not.

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**EXAMPLE 9.10.** Evaluate  $\int_{-100}^{100} \frac{\cos(x)}{\sqrt{x^8 + x^4 + x^2 + 1}} dx$ .

**EXAMPLE 9.11.** (Continue **EXERCISE 9.7**) Evaluate  $\int_{-1}^1 x^6 - x^3 dx$ .

*Fundamental Theorem of Calculus:*

Let  $f$  be integrable over the interval  $[a, b]$ , and suppose there is an antiderivative  $F$  of  $f$  over the interval  $(a, b)$ . Then,

$$\int_a^b f(x) dx = F(b) - F(a).$$

The value of a definite integral is equal to any antiderivative evaluated between the limits of integration: the value at the upper limit minus the value at the lower limit.

**What is a Riemann Sum?** A *Riemann sum* is any sum of the form

$$\sum_{i=1}^n h(x_i^*) \Delta x_i,$$

for some function  $h$ . This algebraic formula does not tell the whole story; it is the end product of a lengthy constructive process.

To completely understand what a *Riemann sum* is, you must understand the story behind the formula. We have some interval  $[a, b]$  of numbers and we subdivide it into  $n \in \mathbb{N}$  subintervals. The length of the  $i^{\text{th}}$  subinterval is  $\Delta x_i$ ,  $i = 1, 2, 3, \dots, n$ . Out of each subinterval, the  $i^{\text{th}}$  one, a number is chosen and labeled  $x_i^*$ .

A *Riemann sum* then, is the sum of the given function  $h$  evaluated at the chosen points,  $h(x_i^*)$ , times the width,  $\Delta x_i$ , of the interval from which the point  $x_i^*$  was chosen from ... and Presto! Changeo! we have *Riemann Sum*:

$$\boxed{\sum_{i=1}^n h(x_i^*) \Delta x_i.}$$

# Solutions to Exercises

**7.1.** The 5<sup>th</sup> interval is  $I_5 = [x_4, x_5]$  and the length of this interval is

$$\Delta x_5 = x_5 - x_4.$$

Exercise 7.1. ■

**7.2.** Why it's the length of the interval  $[a, b]$ . These subintervals subdivide the interval  $[a, b]$ ; therefore, the total length of the subintervals equals the length of the intervals they span.

For the algebraic fanatics out there

$$\begin{aligned}\Delta x_1 + \Delta x_2 + \Delta x_3 + \cdots + \Delta x_n \\ &= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) \\ &= -x_0 + x_n \\ &= x_n - x_0 \\ &= b - a\end{aligned}$$

All numbers in the second line cancel out except the term  $-x_0$  and the term  $x_n$ .

[Exercise 7.2.](#) ■

**7.3.** Why it's the area of the region  $R$ .

Exercise 7.3. ■



**7.4.**  $n$  is the number of rectangles to be constructed, and  $i$  is the index variable that keeps track of the rectangles;  $i = 4$ , for example, corresponds to the 4<sup>th</sup> rectangle.

The quantity,  $\Delta x_i$  is the length of the base of the  $i^{\text{th}}$  rectangle, and  $f(x_i^*)$  is the height of same. Finally,  $f(x_i^*)\Delta x_i$  is the area of the  $i^{\text{th}}$  rectangle.

We add up the areas of the  $n$  rectangles to obtain

$$\sum_{i=1}^n f(x_i^*)\Delta x_i$$

to obtain an approximation of the total area under the graph of  $f$ . This creature is generically referred to as a *Riemann sum*.

**Question.** Some textbooks always subdivide the interval into  $n$  subintervals all of the *same* width. In our development, we don't mind that the subintervals are not of the same width; e.g.  $\Delta x_1$  may be different

in value from  $\Delta x_2$ . Why do you suppose we don't mind having subintervals of possibly different widths? (Formulate an answer within the context of **The Idea of the Solution**.)

Exercise 7.4. ■

**7.5.** Just subdivide the interval  $[0, 0.1]$  into 99 subintervals of equal length, then tack on the interval  $[0.1, 1]$  to make 100 subintervals partitioning the interval  $[0, 1]$ .

You need more details? Say, no.

[Exercise 7.5.](#) ■

**7.6.** This function is nonnegative over the interval specified; therefore, this is the type of situation analyzed in this section.

$$A = \int_{-1}^5 x^4 + x \, dx$$

Exercise 7.6. ■

**7.7.** Are you peeking at the answer? Here's one last hint: Area below the  $x$ -axis is considered *negative* and area above the  $x$ -axis is considered positive.

$$(a) \int_a^b f(x) dx > 0.$$

Given that you have looked at my formulation for part (a), now revise, if needed, your interpretation of (b) before looking.

$$(b) \int_a^b f(x) dx < 0.$$

Finally for part (c). Do you need to reconsider your formulation before looking?

$$(c) \int_a^b f(x) dx = 0.$$

**7.8.** The short answer is “Yes.” The slightly longer answer would be ... the function

$$f(x) = \begin{cases} x^2 & x < 1 \\ 3 - x & x \geq 1 \end{cases}$$

is decreasing on the interval  $(-2, 0)$ , it is increasing on the interval  $(0, 1)$ , and it is decreasing again on the interval  $(1, 2)$ . Why do I make these observations? This is how we argue, in written form, that a function is **piecewise monotone**. We have argued that the interval of interest  $[-2, 2]$  can be partitioned

$$-2 < 0 < 1 < 2$$

such that over each subinterval  $f$  is monotone.

Exercise 7.8. ■

**7.9.** An example of a piecewise continuous function that is *not* piecewise monotone:

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

In fact, this function is everywhere continuous (even at  $x = 0$ ). Yet,  $f$  is *not* piecewise monotone on the interval  $[0, 1]$ , for example. Do you see why? Graph the function and compare it with the definition of **piecewise monotone function**.

*Exercise Notes:* Prove that  $f$  is continuous at  $x = 0$ . *Hint:* Use **squeezing techniques**.

■ Can you give a formal argument that  $f$  is not piecewise monotone?

Exercise 7.9. ■

**7.10.** It is easier to imagine such an example than to write it down mathematically. Here is the result of my imagination.

Define  $f(0) = 0$ . For any  $x \in (0, 1]$ , there is a (unique) natural number  $k$  such that  $1/2^k < x \leq 1/2^{k-1}$ . In this case, define  $f(x)$  to be

$$f(x) = \frac{x}{2^{k-1}}.$$

This is a monotone increasing function on the interval  $[0, 1]$  having infinitely many discontinuities. The points of discontinuity are

$$x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^k}, \dots$$

**Question.** Can you draw the graph of this function? No graphing calculators allowed! *Hint:* For  $k = 1$ , the function  $f$  is defined on the interval  $(1/2, 1]$  by  $f(x) = x$ ; for  $k = 2$ , the function  $f$  is defined on the interval  $(1/4, 1/2]$  by  $f(x) = x/2$ , and so on. See if you can graph it.



This function is not piecewise continuous since  $f$  has infinitely many discontinuities. Piecewise continuous, as defined above specifies only finitely many discontinuities.

**Question.** According to the existence theorem,

$$\int_0^1 f(x) dx \text{ exists and is finite.}$$

Can you calculate the value of this integral? (*Hint:* The integral represents the area under the graph. This area can be calculated by certain already established formula for common geometric objects. Good luck!

DS)

Exercise 7.10. ■

**7.11.** This one is too easy: The index is  $i$ ; its initial value is 1 and its terminal or final value is 5. [Exercise 7.11.](#) ■

**7.12.** Let's take each in turn.

1.  $\sum_{i=1}^4 (i^2 - 2i)$

$$\begin{aligned}\sum_{i=1}^4 (i^2 - 2i) &= (1^2 - 2(1)) + (2^2 - 2(2)) + (3^2 - 2(3)) + (4^2 - 2(4)) \\ &= (-1) + 0 + 2 + 8.\end{aligned}$$

2.  $\sum_{\alpha=1}^4 (5\alpha + 1)$

$$\begin{aligned}\sum_{\alpha=1}^4 (5\alpha + 1) &= (5(1) + 1) + (5(2) + 1) + (5(3) + 1) + (5(4) + 1) \\ &= 6 + 11 + 16 + 21.\end{aligned}$$

## Solutions to Exercises (continued)

$$3. \sum_{j=1}^5 \frac{j}{(j^2+2)}$$

$$\begin{aligned} \sum_{j=1}^5 \frac{j}{(j^2+2)} &= \frac{1}{(1^2+2)} + \frac{2}{(2^2+2)} + \frac{3}{(3^2+2)} + \frac{4}{(4^2+2)} + \frac{5}{(5^2+2)} \\ &= \frac{1}{3} + \frac{1}{3} + \frac{3}{11} + \frac{2}{9} + \frac{5}{27}. \end{aligned}$$

$$4. \sum_{i=4}^7 i^2$$

$$\begin{aligned} \sum_{i=4}^7 i^2 &= 4^2 + 5^2 + 6^2 + 7^2 \\ &= 16 + 25 + 36 + 49 \end{aligned}$$

The last expansion hopefully did not throw you for a loop. The initial value of the index is  $i = 4$ . This sum corresponds to equation (15). The **algorithm** is valid for this kind of sum. Exercise 7.12. ■

**7.13. MEDITATE!**

What do I refer to when I say “the sum of a sum”?

What do I refer to when I say “the sum of the sums”?

Exercise 7.13. ■

**7.14.** You should have proceeded as follows.

*Part 1:*

$$\begin{aligned}
 \sum_{i=1}^{20} (i^2 - 2i) &= \sum_{i=1}^{20} i^2 + \sum_{i=1}^{20} (-2i) && \triangleleft \text{by (2)} \\
 &= \sum_{i=1}^{20} i^2 - 2 \sum_{i=1}^{20} i && \triangleleft \text{by (1)} \\
 &= 2,870 - 2(210) && \triangleleft \text{by (16)} \\
 &= 2,420 && \triangleleft \text{by pencil and paper}
 \end{aligned}$$

*Part 2:* This part is done similarly. I rely on you to complete it correctly if you have not done so already. Use the above solution as an example of the technique and style of solution. Pay attention to proper notation—not only do you want to learn mathematics, but you also want to become mathematically literate.

Use the properties of summation explicitly. Write everything down, just as I have above. Remember, it's easier for you to write the solution than for me to type the solution.

**Quiz.**  $\sum_{\alpha=1}^{20} (3\alpha^2 - 4\alpha) = ??$

(a) 1,750

(b) 6,440

(c) 7,770

(d) 8,330

Exercise 7.14. ■

**7.15.** The idea is to expand the sums and manipulate them in a more traditional way.

*Proof that* 
$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i.$$

We begin with the left-hand side and show it is equal to the right-hand side.

$$\begin{aligned} \sum_{i=1}^n ca_i &= (ca_1) + (ca_2) + \cdots + (ca_n) && \triangleleft \text{by the algorithm} \\ &= c[a_1 + a_2 + \cdots + a_n] && \triangleleft \text{Quiz \#1} \\ &= c \sum_{i=1}^n a_i && \triangleleft \text{by the algorithm} \end{aligned}$$

**Quiz #1.** Which of the following algebraic laws justifies this step:

- |                 |                  |
|-----------------|------------------|
| (a) associative | (b) distributive |
| (c) commutative | (d) n.o.t.       |



## Solutions to Exercises (continued)

$$\textit{Proof of } \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i.$$

We begin with the left-hand side and show it is equal to the right-hand side.

$$\begin{aligned} & \sum_{i=1}^n (a_i + b_i) \\ &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) && \triangleleft \text{by the algorithm} \\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) && \triangleleft \text{Quiz \#2} \\ &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i && \triangleleft \text{by the algorithm} \end{aligned}$$

**Quiz #2.** Two algebraic laws were used in this step. Your assignment, should you decide to accept it, is to name these two.

- (a) distributive and associative    (b) distributive and commutative  
(c) associative and commutative    (d) n.o.t.

*Exercise Notes:* After you have studied the proofs of these two properties, I think you'll agree with me that they are nothing more than commonly known properties of addition and multiplication. When written in the form of the sigma notation they look different and strange, but, nevertheless, they are very simple properties of our number system.

- It would be useful when you use these properties to visualize in your mind the kinds of arithmetic manipulations you are actually doing.

■  
[Exercise 7.15.](#) ■

**7.16.** Just follow the **summation formulas**:

$$\sum_{i=1}^{20} 3 = 20(3) = 60 \quad \triangleleft \text{from (1)}$$

$$\sum_{j=1}^{33} j = \frac{33(34)}{2} = 561 \quad \triangleleft \text{from (2)}$$

$$\sum_{k=1}^{15} k^2 = \frac{15(16)(31)}{6} = 1,240 \quad \triangleleft \text{from (3)}$$

$$\sum_{\alpha=1}^{20} \alpha^3 = \left[ \frac{20(21)}{2} \right]^2 = 44,100. \quad \triangleleft \text{from (4)}$$

Exercise 7.16. ■

**7.17.** The summation notation provides a formula for computing the numbers involved in the sum.

$$1. \sum_{i=1}^{20} 3 = 3 + 3 + 3 + 3 + \cdots + 3 = 60.$$

$$2. \sum_{j=1}^{33} j = 1 + 2 + 3 + 4 + \cdots + 33 = 561.$$

$$3. \sum_{k=1}^{15} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + 15^2 = 1,240.$$

$$4. \sum_{\alpha=1}^{20} \alpha^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + 20^3 = 44,100.$$

Exercise 7.17. ■

**7.18.** You would notice that in the sum

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

all the numbers have the same form:  $a_i$  for some number  $i$ . And this number  $i$ , the *index*, varies from  $i = 1$  to  $i = n$ . So write for the monster the notation:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^n a_i.$$

Or, just to confuse the poor monster, maybe write

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{\text{dps}=1}^n a_{\text{dps}}.$$

Exercise 7.18. ■

**7.19.**

Exercise 7.19. ■



Work out the answer, then check it against the alternatives. Passing score is 1 out of 1.

**Quiz** Which of the following is equal to  $\sum_{j=12}^{20} j^2$ ?

- (a) 3,140      (b) 2,134      (c) 2,364      (d) n.o.t.

*Solution to 3:* Solve the problem using the techniques illustrated for (1). Check your answer below. You will jump the complete solution when you click on the right answer. Do the problem first.

**Quiz** Which of the following is equal to  $\sum_{k=5}^{10} k^3$ ?

- (a) 2,875      (b) 2,925      (c) 2,921      (d) n.o.t.

[Exercise 7.20.](#) ■



**7.21.** Did you write  $\sum_{i=1}^n \sqrt{i + i^2}$ ?

Exercise 7.21. ■

**7.22.** *Ans:*  $\sum_{i=1}^n (1 + f(i))^3.$

Exercise 7.22. ■

**7.23.** *Ans:*  $\sum_{j=1}^k \sin(f(x_j)).$

Did I throw you off with the different letters  $k$  and  $j$

You may have written:  $\sum_{i=1}^k \sin(f(x_i))$ , which is right, or you could have written  $\sum_{i=1}^n \sin(f(x_i))$ , which is technically wrong since I specified that the last number is  $k$  *not*  $n$ . Did I trick you?      [Exercise 7.23.](#) ■

**7.24.** If we have an interval of length 7 units and want to partition it into 2 equal subintervals, the length of each subinterval must be  $7/2 = 3.5$  units.

If we have an interval of length 6 units and we want to divide it into 3 equal parts, the length of each of the three parts will be  $6/3 = 2$  units.

In just the same way, if we have an interval,  $[a, b]$ , of length  $b - a$  and we want to divide this length into  $n$  subintervals of the same length, the width of each of these subintervals must necessarily be  $(b - a)/n$ .

Exercise 7.24. ■

**7.25.** We use the formula (19), here  $[a, b] = [1, 4]$  and  $n = 6$ :

$$\Delta x = \frac{b - a}{n} = \frac{4 - 1}{6} = \boxed{\frac{1}{2}}.$$

That is, the common width of these 6 intervals will be  $\Delta x = 1/2$ .

Exercise 7.25. ■

**7.26.** Each of the partition points are  $\Delta x = (b - a)/n$  apart. This is the “critical” observation. There are a couple of methods that can be used.

*Method 1:*  $x_0 = a$

$$x_1 = x_0 + \Delta x$$

$$x_2 = x_1 + \Delta x$$

$$x_3 = x_2 + \Delta x$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = x_{i-1} + \Delta x$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = x_{n-1} + \Delta x$$

This is a nice computation method of calculating the partition points one at a time. The other method is just a general formula for the calculation of these points.

*Method 2:* For each  $i$ , the value of the  $i^{\text{th}}$  node is

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n. \quad (\text{A-1})$$

**Question.** Do you see where this formula comes from? *Do not leave until you understand!*

[Exercise 7.26.](#) ■

**7.27.** The interval is

$$[a, b] = [1, 4].$$

we want to subdivide this interval into  $n = 6$  subintervals of equal length. That common length is

$$\Delta x = \frac{b - a}{n} = \frac{4 - 1}{6} = .5$$

The partition points or nodes can easily be calculated using **Method 1**:

$$x_0 = a = 1$$

$$x_1 = x_0 + \Delta x = 1.0 + .5 = 1.5$$

$$x_2 = x_1 + \Delta x = 1.5 + .5 = 2$$

$$x_3 = x_2 + \Delta x = 2.0 + .5 = 2.5$$

$$x_4 = x_3 + \Delta x = 2.5 + .5 = 3$$

$$x_5 = x_4 + \Delta x = 3.0 + .5 = 3.5$$

$$x_6 = x_5 + \Delta x = 3.5 + .5 = 4$$

The direct formula, **Method2**, can be used as well. For example, if we wanted to compute  $x_3$  we obtain from **(A-1)**:

$$x_3 = x_0 + 3 \Delta x = 1 + 3(.5) = 2.5$$





**7.28.** The value of the common width of these 8 intervals is

$$\Delta x = \frac{b - a}{n} = \frac{5 - (-1)}{8} = \frac{6}{8} = \frac{3}{4}$$

Using the **direct formula**, we obtain,

$$x_i = x_0 + i\Delta x = -1 + i\frac{3}{4}, \quad i = 1, 2, 3, \dots, 8$$

You can evaluate this formula for the different values of the index  $i$  to obtain the numerical results.

**Quiz.** What is the value of  $x_5$ , the fifth node in the partition?

(a) 2.0

(b) 2.75

(c) 3.5

(d) 4.25

[Exercise 7.28.](#) ■

**7.29.** YOU are supposed to help ME! As I am rather self-reliant, I'll help myself.

The left-hand endpoint is obtained from the formula

$$x_i = 4 + \frac{4i}{5} \quad i = 0, 1, 2, 3, \dots, 20$$


by putting  $i = 0$ . Using my hand-held graphing calculator, I obtain

$$a = x_0 = 4.$$

You can obtain the right-hand endpoint is obtained from the above formula by putting  $i = 20$ . Thus,

$$b = x_{20} = 4 + \frac{4(20)}{5} = 20$$

*Answer:* The interval is  $\boxed{[4, 20]}$ .

Thank you! 

Exercise 7.29. ■

**7.30.** Here is an outline of what you should have done; your solution should have been as complete as EXAMPLE 7.7

Begin by subdividing the interval  $[a, b] = [1, 4]$  into  $n$  equal subintervals.

*Step 1:* Calculate  $\Delta x$ .

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}$$

*Step 2:* Calculate the partition points (using the techniques of EXERCISE 7.26 or EXERCISE 7.27).

*Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = 1$$

$$x_1 = 1 + (1)(3/n) = 1 + 3/n$$

$$x_2 = 1 + (2)(3/n) = 1 + 6/n$$

Solutions to Exercises (continued)

$$x_3 = 1 + (3)(3/n) = 1 + 9/n$$

$$x_4 = 1 + (4)(3/n) = 1 + 12/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = 1 + (i)(3/n) = 1 + 3i/n \tag{A-2}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = 1 + (n)(3/n) = 1 + 3 = 4 \dots \text{ and done!}$$

*Step 3:* Choose the intermediate points  $x_i^*$ : Say the right-hand endpoints of each interval. From (A-2) we obtain:

$$x_i^* = x_i = 1 + \frac{3i}{n} \quad i = 1, 2, 3, \dots, n.$$

(**Recall:** the  $i^{\text{th}}$  interval is  $[x_{i-1}, x_i]$  and so the right-hand endpoint of this interval is  $x_i$ .)

**Summary.**  $x_i^* = 1 + \frac{3i}{n}$  and  $\Delta x = \frac{2}{n}$ .

*Step 4:* Calculate **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(1 + 3i/n) \frac{3}{n} \\ &= \sum_{i=1}^n 5 \frac{3}{n} \\ &= \sum_{i=1}^n \frac{15}{n} \\ &= (n) \frac{15}{n} \quad \triangleleft \text{by (1) above} \\ &= 15 \end{aligned}$$

Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = 15. \tag{A-3}$$

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (A-3):

$$\int_1^4 5 dx = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 15 = 15,$$

or,

$$\boxed{\int_1^4 5 dx = 15.}$$

Once again we obtain the expected result.

Exercise 7.30. ■

**7.31.** Begin by subdividing the interval  $[a, b] = [0, b]$  into  $n$  equal subintervals.

*Step 1:* Calculate  $\Delta x$ .

$$\Delta x = \frac{b - a}{n} = \frac{b}{n}$$

*Step 2:* Calculate the partition points (using the techniques of **EXERCISE 7.26** or **EXERCISE 7.27**).

*Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = 0$$

$$x_1 = (1)(b/n) = b/n$$

$$x_2 = (2)(b/n) = 2b/n$$

$$x_3 = (3)(b/n) = 3b/n$$



Solutions to Exercises (continued)

$$x_4 = (4)(b/n) = 4b/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = (i)(b/n) = ib/n \tag{A-4}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = (n)(b/n) = b \dots \text{ and done!}$$

*Step 3:* Choose the intermediate points  $x_i^*$ : Say the right-hand endpoints of each interval. From (A-4) we obtain:

$$x_i^* = x_i = \frac{ib}{n} \quad i = 1, 2, 3, \dots, n.$$

**Summary.**  $x_i^* = \frac{ib}{n}$  and  $\Delta x = \frac{b}{n}$ .

Recall we are integrating the function  $f(x) = h$ .

Step 4: Calculate **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(ib/n) \frac{b}{n} \\ &= \sum_{i=1}^n h \frac{b}{n} \\ &= \sum_{i=1}^n \frac{bh}{n} \\ &= (n) \frac{bh}{n} \quad \triangleleft \text{by (1) above} \\ &= bh \end{aligned}$$

Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = bh. \tag{A-5}$$

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (A-5):

$$\int_0^b h \, dx = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} bh = bh,$$

or,

$$\boxed{\int_0^b h \, dx = bh.}$$

Once again we obtain the expected result—that's more on that particular construction than you really wanted to see. [Exercise 7.31.](#) ■

**7.32.** The function  $f$  that we are integrating is  $f(x) = 4x$ . This function is continuous over the interval  $[0, 5]$ ; therefore, by the *Existence Theorem*, **Theorem 7.4**, the function  $f(x) = 4x$  is integrable over the interval  $[0, 5]$ ; that is,

$$\int_0^5 4x \, dx \text{ exists and is finite.}$$

That being the case, from the constructive **definition**, no matter what the choice of the intermediate points,  $x_i^*$ , the corresponding Riemann sums always tend to the same value (the value of the integral).

If we choose the intermediate points to be the right-hand endpoints, since  $f$  is integrable, the corresponding Riemann sums *must* tend to the value of the integral. We showed in **EXAMPLE 7.8** that those Riemann sums tend to 50. Therefore 50 must be the value of the integral

$$\int_0^5 4x \, dx = 50,$$

and any other choice of intermediate points would yield Riemann sums that would necessarily have to tend to 50 as well.

That clear?

Exercise 7.32. ■

**7.33.** If you replace 4 by  $-4$  in the demonstration given in **EXAMPLE 7.8**, and follow the change in the calculations, you will see that

$$\int_0^5 -4x \, dx = -50$$

Or, writing it differently to suggest a general principle,

$$\int_0^5 -4x \, dx = - \int_0^5 4x \, dx$$

Exercise 7.33. ■

**7.34.** Begin by subdividing the interval  $[a, b] = [0, 2]$  into  $n$  equal subintervals.

*Step 1:* Calculate  $\Delta x$ .

$$\Delta x = \frac{b - a}{n} = \frac{2}{n}$$

*Step 2:* Calculate the partition points (using the techniques of **EXERCISE 7.26** or **EXERCISE 7.27**).

*Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = 0$$

$$x_1 = (1)(2/n) = 2/n$$

$$x_2 = (2)(2/n) = 4/n$$

$$x_3 = (3)(2/n) = 6/n$$

Solutions to Exercises (continued)

$$x_4 = (4)(2/n) = 8/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = (i)(2/n) = 2i/n \tag{A-6}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = (n)(2/n) = 2 \dots \text{ and done!}$$

*Step 3:* Choose the intermediate points  $x_i^*$ : Say the right-hand endpoints of each interval. From (A-6) we obtain:

$$x_i^* = x_i = \frac{2i}{n} \quad i = 1, 2, 3, \dots, n.$$

**Summary.**  $x_i^* = \frac{2i}{n}$  and  $\Delta x = \frac{2}{n}$ .

Recall we are integrating the function  $f(x) = 3x + 1$ ; consequently,

$$f(2i/n) = 3\frac{2i}{n} + 1 = \frac{6i}{n} + 1. \tag{A-7}$$



This calculation is needed in the next step.

*Step 4:* Calculate **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(2i/n) \frac{2}{n} \\ &= \sum_{i=1}^n \left( \frac{6i}{n} + 1 \right) \frac{2}{n} &< \text{from (A-7)} \\ &= \sum_{i=1}^n \left( \frac{12i}{n^2} + \frac{2}{n} \right) \\ &= \sum_{i=1}^n \frac{12i}{n^2} + \sum_{i=1}^n \frac{2}{n} = \frac{12}{n^2} \sum_{i=1}^n i + \sum_{i=1}^n \frac{2}{n} &< \text{by (2)} \\ &= \frac{12}{n^2} \frac{n(n+1)}{2} + (n) \frac{2}{n} &< \text{by (2) \& (1)} \end{aligned}$$

Solutions to Exercises (continued)

$$= 6 \frac{n+1}{n} + 2$$

Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = 6 \frac{n+1}{n} + 2. \quad (\text{A-8})$$

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (A-8):

$$\begin{aligned} \int_0^5 4x \, dx &= \lim_{n \rightarrow \infty} S_n \\ &= \lim_{n \rightarrow \infty} 6 \frac{n+1}{n} + 2 \\ &= 6 \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) + \lim_{n \rightarrow \infty} 2 \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

or,

$$\int_0^2 3x + 1 \, dx = 8.$$

*Example Notes:* The graph of  $f$ ,  $0 \leq x \leq 2$ , is a straight line going from the point  $(0, 1)$  to the point  $(2, 7)$ . If you draw this graph and look at the region under the graph and above the  $x$ -axis, you will see a *trapezoid*. The area of a trapezoid is

$$A = \frac{1}{2}(h_1 + h_2)b,$$

where  $h_1$ , and  $h_2$  are the two altitudes of the trapezoid and  $b$  is the base of the trapezoid. In our case  $h_1 = f(0) = 1$ ,  $h_2 = f(2) = 7$ , and  $b = 2$ . According to the trapezoid formula,

$$A = \frac{1}{2}(1 + 7)(2) = 8,$$

that checks! ■

**7.35.** Subdivide the interval  $[a, b] = [-1, 2]$  into  $n$  equal subintervals.

*Step 1:* Calculate  $\Delta x$ .

$$\Delta x = \frac{b - a}{n} = \frac{3}{n}$$

*Step 2:* Calculate the partition points (using the techniques of **EXERCISE 7.26** or **EXERCISE 7.27**).

*Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = -1$$

$$x_1 = -1 + (1)(3/n) = -1 + 3/n$$

$$x_2 = -1 + (2)(3/n) = -1 + 6/n$$

$$x_3 = -1 + (3)(3/n) = -1 + 9/n$$

Solutions to Exercises (continued)

$$x_4 = -1 + (4)(3/n) = -1 + 12/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = -1 + (i)(3/n) = -1 + 3i/n \tag{A-9}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = -1 + (n)(3/n) = 2 \dots \text{ and done!}$$

*Step 3:* Choose the intermediate points  $x_i^*$ : Say the right-hand end-points of each interval. From (A-9) we obtain:

$$x_i^* = x_i = -1 + \frac{3i}{n} \quad i = 1, 2, 3, \dots, n.$$

**Summary.**  $x_i^* = -1 + \frac{3i}{n}$  and  $\Delta x = \frac{3}{n}$ .

Recall we are integrating the function  $f(x) = 3x^2 - 2x$ ; consequently,

$$\begin{aligned} f(-1 + 3i/n) &= 3(-1 + 3i/n)^2 - 2(-1 + 3i/n) \\ &= 3 \left( 1 - \frac{6i}{n} + \frac{9i^2}{n^2} \right) + 2 - \frac{6i}{n} \\ &= 5 - \frac{24i}{n} + \frac{27i^2}{n^2} \end{aligned} \tag{A-10}$$

This calculation is needed in the next step.

*Step 4:* Calculate **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(-1 + 3i/n) \frac{3}{n} \\ &= \sum_{i=1}^n \left( 5 - \frac{24i}{n} + \frac{27i^2}{n^2} \right) \frac{3}{n} \end{aligned} \quad \triangleleft \text{from (A-10)}$$

## Solutions to Exercises (continued)

$$\begin{aligned}
&= \sum_{i=1}^n \frac{15}{n} - \sum_{i=1}^n \frac{72i}{n^2} + \sum_{i=1}^n \frac{81i^2}{n^3} \\
&= \sum_{i=1}^n \frac{15}{n} - \frac{72}{n^2} \sum_{i=1}^n i + \frac{81}{n^3} \sum_{i=1}^n i^2 \quad \triangleleft \text{by (2)}
\end{aligned}$$

We now invoke the **summation formulas**:

$$\begin{aligned}
S_n &= (n) \frac{15}{n} - \frac{72}{n^2} \frac{n(n+1)}{2} + \frac{81}{n^3} \frac{n(n+1)(2n+1)}{6} \quad \triangleleft \text{by } \begin{cases} (1), (2) \\ \text{and } (3) \end{cases} \\
&= 15 - 36 \frac{n+1}{n} + \frac{27}{2} \frac{n+1}{n} \frac{2n+1}{n}
\end{aligned}$$

Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = 15 - 36 \frac{n+1}{n} + \frac{27}{2} \frac{n+1}{n} \frac{2n+1}{n}. \quad (\text{A-11})$$

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (A-11):

$$\begin{aligned}\int_{-1}^2 3x^2 - 2x \, dx &= \lim_{n \rightarrow \infty} S_n \\ &= \lim_{n \rightarrow \infty} \left( 15 - 36 \frac{n+1}{n} + \frac{27}{2} \frac{n+1}{n} \frac{2n+1}{n} \right) \\ &= 15 - 36 \lim_{n \rightarrow \infty} \frac{n+1}{n} + \frac{27}{2} \lim_{n \rightarrow \infty} \frac{2n+1}{n} \\ &= 15 - 36 + \frac{27}{2}(2) \\ &= 6\end{aligned}$$

or,

$$\boxed{\int_{-1}^2 3x^2 - 2x \, dx = 6.}$$



**8.1.** Let  $F$  be *any* antiderivative of  $f$ , then  $F + C$  is also an antiderivative of  $f$ . Thus,

$$\begin{aligned}\int_a^b f(x) dx &= (F(b) + C) - (F(a) + C) \\ &= F(b) + C - F(a) - C \\ &= F(b) - F(a).\end{aligned}$$

The inclusion of the constant  $C$  did not contribute to the value of the integral, all it did was create a more complicated expression to evaluate. Exercise 8.1. ■

**9.1.** Routine:

$$\begin{aligned}\int_{-2}^2 x^2 - 5x^5 dx &= \left( \frac{1}{3}x^3 - \frac{5}{6}x^6 \right) \Big|_{-2}^2 \\ &= \frac{1}{3}(8 + 8) - \frac{5}{6}(0) \\ &= \boxed{\frac{16}{3}}.\end{aligned}$$

In doing this problem, I hope you concentrated on using the properties to shorten the numerical calculations *as well as* using *good techniques!*

Exercise 9.1. ■

**9.2.** Hopefully, you are getting the hang of it.

$$\begin{aligned}\int_2^3 3x^3 + 2x^2 dx &= \left( \frac{3}{4}x^4 + \frac{2}{3}x^3 \right) \Big|_2^3 \\ &= \frac{3}{4}(81 - 16) + \frac{2}{3}(27 - 8) \\ &= \frac{3}{4}65 + \frac{2}{3}19 \\ &= \boxed{\frac{793}{12}}\end{aligned}$$

Exercise 9.2. ■

**9.3.** First, solve the associate *indefinite integral*: We will be applying the **Power Rule**, so we put  $u = x^2 + 1$ , and  $du = 2x dx$ .

*Calculate Indefinite Integral:*

$$\begin{aligned}\int \frac{x}{(x^2 + 1)^3} dx &= \int (x^2 + 1)^{-3} x dx && \triangleleft \text{prepare for Power} \\ &= \frac{1}{2} \int (x^2 + 1)^{-3} 2x dx && \triangleleft \text{apply fudge factor} \\ &= \frac{1}{2} \int u^{-3} du && \triangleleft \text{substitution} \\ &= \frac{1}{2} \frac{u^{-2}}{-2} + C \\ &= -\frac{1}{4} (x^2 + 1)^{-2} + C\end{aligned}$$

*Calculate Definite Integral:*

$$\begin{aligned}\int \frac{x}{(x^2 + 1)^3} dx &= -\frac{1}{4} (x^2 + 1)^{-2} \Big|_{-2}^0 \\ &= -\frac{1}{4} (5^{-2} - 1) \quad \triangleleft \text{EBLO Tricks} \\ &= -\frac{1}{4} \left( -\frac{24}{25} \right) \\ &= \boxed{\frac{6}{25}}.\end{aligned}$$

Hope that you made the same moves on this one.

Exercise 9.3. ■

**9.4.** We use the **Additive Property** of the limits of integration.

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \quad \triangleleft (10)$$

but,

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad \triangleleft \text{Additive Prop.}$$

therefore,

$$\int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx \quad (\text{A-12})$$

Solve the equation (A-12) algebraically for  $\int_0^a f(x) dx$  to obtain,

$$\boxed{\int_0^a f(x) dx = \int_{-a}^0 f(x) dx.}$$

These techniques were contained in the **proof** of **Theorem 9.1**. One of the uses of proof construction is that they, sometimes, reveal valuable

## Solutions to Exercises (continued)

techniques not normally seen in ordinary problem solving (in *Calculus*.

[Exercise 9.4.](#) ■

**9.5.** We are indeed integrating an even function over a symmetric interval; therefore we can take advantage of symmetry — if we wish.

$$\begin{aligned}\int_{-1}^1 x^4 - 2x^2 - 1 \, dx &= 2 \int_0^1 x^4 - 2x^2 - 1 \, dx \\ &= 2 \left. \frac{x^5}{5} - \frac{2x^3}{3} - x \right|_0^1 \\ &= 2 \left( \frac{1}{5} - \frac{2}{3} - 1 \right) \\ &= \boxed{-\frac{44}{15}}.\end{aligned}$$

Exercise 9.5. ■



**9.6.** We are indeed integrating an even function over a symmetric interval; therefore we can take advantage of symmetry — if we wish. One of the properties of the cosine function is that  $\cos(-x) = \cos(x)$ . This is precisely the definition of an **even function**.

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \cos(x) dx &= 2 \int_0^{\pi/4} \cos(x) dx \\ &= 2 \sin(x) \Big|_0^{\pi/4} \\ &= 2 \sin\left(\frac{\pi}{4}\right) \\ &= \sqrt{2}\end{aligned}$$

Exercise 9.6. ■

**9.7.** I hope you did not try to use the symmetry property (10), because the function is not even, even though we are integrating over a symmetric interval. If you went merrily on your way, doubling the integral and integrating from 0 to 1, then you have not paid attention to the **Keys to Success**.

*The Problem:*

$$\int_{-1}^1 x^6 - x^3 dx.$$

We can solve this (simple) integral in the usual way. But here is an idea: The first term is even and so we can take advantage of its symmetry.

*Evaluation:*

$$\begin{aligned}\int_{-1}^1 x^6 - x^3 dx &= \int_{-1}^1 x^6 dx - \int_{-1}^1 x^3 dx \\ &= 2 \int_0^1 x^6 dx - \int_{-1}^1 x^3 dx \\ &= 2 \frac{1}{7} x^7 \Big|_0^1 - \frac{1}{4} x^4 \Big|_{-1}^1 \\ &= \frac{2}{7} - \frac{1}{4} (1 - (-1)^4) \\ &= \frac{2}{7} - \frac{1}{4} (1 - 1) \\ &= \boxed{\frac{2}{7}}.\end{aligned}$$

This calculation was stretched out a bit. After the discussion on **odd functions**, this calculation can be reduced considerably. See the **example** below.

[Exercise 9.7.](#) ■

**9.8.** We use the **Additive Property** of the limits of integration.

$$\int_{-a}^a f(x) dx = 0 \quad \triangleleft (12)$$

but,

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad \triangleleft \text{Additive Prop.}$$

therefore,

$$\int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 0.$$

Now, solve algebraically the equation

$$\int_{-a}^0 f(x) dx + \int_0^a f(x) dx = 0$$

for  $\int_0^a f(x) dx$  to obtain,

$$\int_0^a f(x) dx = - \int_{-a}^0 f(x) dx.$$

These techniques were contained in the **proof** of **Theorem 9.2**. One of the uses of proof construction is that they, sometimes, reveal valuable techniques not normally seen in ordinary problem solving (in *Calculus*.

Exercise 9.8. ■

# Solutions to Examples

**7.1.** The given function is nonnegative over the given interval; therefore, by the **Solution to the Fundamental Problem**, the area under the graph is given by

$$A = \int_0^1 x^2 dx$$

Example 7.1. ■

**7.2.** An examination of the function will yield the fact that  $f(x) \geq 0$  over the interval  $[1, 4]$ ; therefore,

$$A = \int_1^4 3x^3 + x \, dx$$

Example 7.2. ■

**7.3.** The reasoning would go as follows.

The function is not continuous at  $x = 1$ . Indeed,

$$\begin{aligned}\lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} x^3 = 1 \\ \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} 3 - x = 2\end{aligned}$$

Thus,

$$\lim_{x \rightarrow 1^-} f(x) = 1 \neq 2 = \lim_{x \rightarrow 1^+} f(x),$$

that is, the left-hand limit does not equal the right-hand limit; therefore, the limit does not exist and  $f$  cannot be continuous there.

Is  $f$  discontinuous at any other point in the interval  $[-2, 2]$ ? We reason as follows.

On the interval  $[-2, 1)$ ,  $f(x) = x^2$ . Over this interval,  $f$  is a second-degree polynomial function. A polynomial function is continuous over every interval, so we conclude that  $f$  is continuous over the interval  $[-2, 1)$ .



## Solutions to Examples (continued)

On the interval  $(1, 2]$ ,  $f(x) = 3 - x$ . Over this interval,  $f$  is a first-degree polynomial function. A polynomial function is continuous over every interval, so we conclude that  $f$  is continuous over the interval  $(1, 2]$ .

We have argued that  $f$  has one and only one discontinuity; hence,  $f$  is **piecewise continuous**. Example 7.3. ■

**7.4.** For item (1), we are summing a constant. This calls for the use of formula (1). Here,  $n = 40$  and  $c = 2$ ; thus,

$$\sum_{i=1}^n c = nc \quad \triangleleft \text{the formula}$$

$$\sum_{i=1}^{40} 2 = (40)2 = 80, \quad \triangleleft \text{the problem, take } n = 40 \text{ \& } c = 2.$$

All we are doing here is adding 2 together with itself 40 times.

*Solution to (2):* This is a direct application of formula (4) above:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \triangleleft \text{the formula}$$

$$\begin{aligned} \sum_{i=1}^{20} i &= \frac{20(21)}{2} && \triangleleft \text{the problem. Take } n = 20 \\ &= 210 \end{aligned}$$

*Solution to (3):* This is a direct application of formula (3) above:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \triangleleft \text{the formula}$$

$$\begin{aligned} \sum_{i=1}^{10} i^2 &= \frac{20(21)(41)}{6} && \triangleleft \text{the problem. Take } n = 10 \\ &= 2,870 \end{aligned}$$

*Solution to (4):* This is a direct application of formula (4) above:

$$\sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \quad \triangleleft \text{the formula}$$

$$\begin{aligned} \sum_{i=1}^{15} i^3 &= \left[ \frac{15(16)}{2} \right]^2 && \triangleleft \text{the problem. Take } n = 15 \\ &= 14,400 \end{aligned}$$

**7.5.** These problems are similar to **EXERCISE 7.14**, but now we have the summation formulas to help us.

*Solution to 1:*

$$\begin{aligned}\sum_{i=1}^{10} i(3i + 2) &= \sum_{i=1}^{10} (3i^2 + 2i) \\ &= 3 \sum_{i=1}^{10} i^2 + 2 \sum_{i=1}^{10} i && \triangleleft \text{properties of summation} \\ &= 3 \frac{10(11)(21)}{6} + 2 \frac{10(11)}{2} && \triangleleft \text{sum formula: (3) \& (2)} \\ &= 1,115 + 110 = \boxed{1,225}\end{aligned}$$

## Solutions to Examples (continued)

*Solution to 2:*

$$\begin{aligned} & \sum_{j=1}^{15} (4j^3 - 9j^2 + 2) \\ &= 4 \sum_{j=1}^{15} j^3 - 9 \sum_{j=1}^{15} j^2 + \sum_{j=1}^{15} 2 && \triangleleft \text{properties of summation} \\ &= 4 \left[ \frac{15(16)}{2} \right]^2 - 9 \frac{15(16)}{2} + (15)2 && \triangleleft \text{sum formulas: (1), (3), (4)} \\ &= \boxed{13,350} && \triangleleft \text{calculator!} \end{aligned}$$

The final part is an abstract version of the others. Rather than having a definite numerical value as the upper limit of the index, we have a symbolic upper limit  $n$ .

## Solutions to Examples (continued)

*Solution to 3:*

$$\sum_{k=1}^n (3k^2 - k)$$

$$= 3 \sum_{k=1}^n k^2 - \sum_{k=1}^n k$$

◁ properties of summation

$$= 3 \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

◁ sum formulas: (2)–(3)

$$= \frac{n(n+1)}{2} (2n+1-1)$$

◁ cancel and factor!

$$= \frac{n(n+1)}{2} (2n)$$

◁ subtract!

$$= \boxed{n^2(n+1)}$$

◁ cancel and combine!

That looks pretty good!

Example 7.5. ■

**7.6.** The pattern is very strong—looks like powers of 2 to me. Here is the series:

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n \quad (\text{S-1})$$

*First Try:* An obvious choice is  $\sum_{i=1}^n 2^i$ , but that would not expand to (S-1). By the **algorithm** for expanding the sigma notation, we have

$$\sum_{i=1}^n 2^i = 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n.$$

The first term of (S-1) is missing.

*A Correct Try:* The missing term is 1 which equals  $2^0$ , to write 1 as a power of 2. Thus,

$$\begin{aligned} \sum_{i=0}^n 2^i &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n \\ &= 1 + 2 + 2^2 + 2^3 + \cdots + 2^n \end{aligned}$$

## Solutions to Examples (continued)

(Yes, we can start the index at  $i = 0$ .)

*Another Try:* Another way of writing (S-1) would be

$$\begin{aligned}\sum_{i=1}^{n+1} 2^{i-1} &= 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + \dots + 2^{(n+1)-1} \\ &= 1 + 2 + 2^2 + 2^3 + \dots + 2^n\end{aligned}$$

There is *infinitely many* ways of expressing any given sum using the sigma notation. Sometimes we need to start the index at 1, so my first correct solution would not do. The second correct solution would do the trick.

**Question.** How would you set up the sigma notation if you were required to start at  $i = 3$ ?

Example 7.6. ■



**7.7.** Begin by subdividing the interval  $[a, b] = [0, 2]$  into  $n$  equal subintervals.

*Step 1:* The first task is to calculate the common length of these intervals:

$$\Delta x = \frac{b - a}{n} = \frac{2}{n} \quad (\text{S-2})$$

*Step 2:* The next problem is to calculate the partition points; the techniques brought out earlier in **EXERCISE 7.26** and **EXERCISE 7.27** will be useful here.

I'll use *Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = 0$$

$$x_1 = (1)(2/n) = 2/n$$

$$x_2 = (2)(2/n) = 4/n$$

## Solutions to Examples (continued)

$$x_3 = (3)(2/n) = 6/n$$

$$x_4 = (4)(2/n) = 8/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = (i)(2/n) = 2i/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = (n)(2/n) = 2 \dots \text{and done!}$$

*Step 3:* The next step in the constructive process is to choose the intermediate points  $x_i^*$  from each of the intervals. In this case, I'll choose the intermediate points to be the *right-hand endpoint* of each interval; i.e. choose

$$x_i^* = x_i = \frac{2i}{n} \quad i = 1, 2, 3, \dots, n.$$

**Summary.**  $x_i^* = \frac{2i}{n}$  and  $\Delta x = \frac{2}{n}$ .

*Step 4:* This is the information we need to form the Riemann Sums. The next step in the process is to build the **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(2i/n) \frac{2}{n} \\ &= \sum_{i=1}^n 3 \frac{2}{n} \\ &= \sum_{i=1}^n \frac{6}{n} \\ &= (n) \frac{6}{n} \quad \triangleleft \text{by (1) above} \\ &= 6 \end{aligned}$$

Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = 6. \quad (\text{S-3})$$

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (S-3):

$$\int_0^2 3 dx = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 6 = 6,$$

or,

$$\boxed{\int_0^2 3 dx = 6.}$$

*Example Notes:* Normally the value of the  $n^{\text{th}}$  Riemann Sum will depend on  $n$ . Not in this case. The  $n^{\text{th}}$  Riemann Sum in (S-3) does not depend on  $n$ . This is because we are using rectangles to approximate a rectangular region; naturally, the approximation turns out to be *exact!* The value of  $S_n = 6$  is actually the exact area under the curve. Subsequent examples and exercises will be more interesting in this regard.

## Solutions to Examples (continued)

■ This example shows that the definite integral yields the expected answer. ■

Example 7.7. ■

**7.8.** Begin by subdividing the interval  $[a, b] = [0, 5]$  into  $n$  equal subintervals.

*Step 1:* Calculate  $\Delta x$ .

$$\Delta x = \frac{b - a}{n} = \frac{5}{n}$$

*Step 2:* Calculate the partition points (using the techniques of **EXERCISE 7.26** or **EXERCISE 7.27**).

*Method 2* of 7.26:

$$x_i = x_0 + i \Delta x, \quad i = 1, 2, 3, \dots, n.$$

$$x_0 = 0$$

$$x_1 = (1)(5/n) = 5/n$$

$$x_2 = (2)(5/n) = 10/n$$

$$x_3 = (3)(5/n) = 15/n$$

Solutions to Examples (continued)

$$x_4 = (4)(5/n) = 20/n$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_i = (i)(5/n) = 5i/n \tag{S-4}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = (n)(5/n) = 5 \dots \text{ and done!}$$

*Step 3:* Choose the intermediate points  $x_i^*$ : Say the right-hand endpoints of each interval. From (S-4) we obtain:

$$x_i^* = x_i = \frac{5i}{n} \quad i = 1, 2, 3, \dots, n.$$

**Summary.**  $x_i^* = \frac{5i}{n}$  and  $\Delta x = \frac{5}{n}$ .

Recall we are integrating the function  $f(x) = 4x$ ; consequently,

$$f(5i/n) = 4\frac{5i}{n} = \frac{20i}{n}. \tag{S-5}$$

This calculation is needed in the next step.

*Step 4:* Calculate **Riemann Sums**:

$$\begin{aligned} S_n &= \sum_{i=1}^n f(x_i^*) \Delta x \\ &= \sum_{i=1}^n f(4i/n) \frac{5}{n} \\ &= \sum_{i=1}^n \frac{20i}{n} \frac{5}{n} && \triangleleft \text{from (S-5)} \\ &= \sum_{i=1}^n \frac{100i}{n^2} \\ &= \frac{100}{n^2} \sum_{i=1}^n i = \frac{100}{n^2} \frac{n(n+1)}{2} && \triangleleft \text{by (2) above} \\ &= \frac{50(n+1)}{n}. \end{aligned}$$



Thus, the  $n^{\text{th}}$  Riemann Sum is

$$S_n = \frac{50(n+1)}{n}. \quad (\text{S-6})$$

Note that the value of the  $n^{\text{th}}$  Riemann sum does depend on the number of subdivisions (unlike the simpler previous examples and exercises).

Consequently, the final step is

*Step 5:* Calculate the limit of the Riemann sum approximation (S-6):

$$\begin{aligned} \int_0^5 4x \, dx &= \lim_{n \rightarrow \infty} S_n \\ &= \lim_{n \rightarrow \infty} \frac{50(n+1)}{n} \\ &= 50 \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right) \\ &= 50 \end{aligned}$$

or,

$$\int_0^5 4x \, dx = 50.$$

*Example Notes:* This is precisely the result we were looking for. The area of the triangular region is  $A = \frac{1}{2}bh$ ; in this case,  $b = 5$  and  $h = 20$ . Thus, by the formula for the area of a triangle, the area of the region should be  $A = \frac{1}{2}(5)(20) = 50$ —same result, Hurray! ■

Example 7.8. ■

**8.1.** This problem represents a typical calculation.

$$\begin{aligned}\int_2^3 x^2 dx &= \left. \frac{x^3}{3} \right|_2^3 &< \text{Power Rule} \\ &= \frac{3^3}{3} - \frac{2^3}{3} \\ &= \frac{19}{3}.\end{aligned}$$

Since the function  $f(x) = x^2$  is nonnegative over the interval  $[2, 3]$ , the interpretation of the integral

$$\int_2^3 x^2 dx = \frac{19}{3},$$

is that it is the *area* under the graph of  $f$ .

Example 8.1. ■

**8.2.** We proceed as follows:

$$\begin{aligned}\int_0^4 x\sqrt{x} dx &= \int_0^4 x^{3/2} dx \\ &= \frac{2}{5} x^{5/2} \Big|_0^4 &< \text{Power Rule} \\ &= \frac{2}{5} (4^{5/2} - 0) \\ &= \frac{2}{5} (32) \\ &= \frac{64}{5}\end{aligned}$$

Since the function  $f(x) = x\sqrt{x}$  is nonnegative over the interval  $[0, 4]$ , the value of the integral can be interpreted as the *area* under the graph of  $f$  over the interval  $[0, 4]$ . Example 8.2. ■

**8.3.** We use standard procedures.

$$\begin{aligned} & \int_{-2}^{-1} x^4 - 2x \, dx \\ &= \frac{1}{5} x^5 \Big|_{-2}^{-1} - x^2 \Big|_{-2}^{-1} && \triangleleft \text{Power Rule} \\ &= \frac{1}{5} \left( (-1)^5 - (-2)^5 \right) - \left( (-1)^2 - (-2)^2 \right) && \triangleleft \text{Eval. Rule} \\ &= \frac{1}{5} \left( (-1) - (-32) \right) - (1 - 4) \\ &= \frac{31}{5} + 3 \\ &= \frac{46}{5} = 9 \frac{1}{5}. \end{aligned}$$

The value of this integral is interpreted as the *area* under the graph of  $f$ . Example 8.3. ■

**9.1.** We do this two ways the first way, not using the **Homogeneity Property**, the second method using it.

*Method 1:*

$$\begin{aligned}\int_2^3 x^3 dx &= \left. \frac{x^4}{4} \right|_2^3 \\ &= \left( \frac{3^4}{4} - \frac{2^4}{4} \right) \\ &= \left( \frac{81}{4} - \frac{16}{4} \right) \\ &= \frac{65}{4}\end{aligned}$$

That didn't seem too bad, but we carried the numerical constant  $1/4$  throughout our calculation: I had to type 4 in the denominator of every expression throughout the development of the answer.

*Method 2:*

$$\begin{aligned}\int_2^3 x^3 dx &= \frac{x^4}{4} \Big|_2^3 \\ &= \frac{1}{4} x^4 \Big|_2^3 \quad \triangleleft \text{EBLO Homogen.} \\ &= \frac{1}{4} (3^4 - 2^4) \\ &= \frac{65}{4}\end{aligned}$$

Here, I *factored* the multiplicative constant  $1/4$  out and just evaluated the remaining functional part. I didn't figure the  $1/4$  into every expression, as a result, less typing, and a cleaner, easier calculation.

Example 9.1. ■

**9.2.** Let's again illustrate two ways. The indefinite integral is

$$\int x^3 - x^2 dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + C.$$



Solutions to Examples (continued)

*Method 1:*

$$\begin{aligned} & \int_{-1}^1 x^3 - x^2 dx \\ &= \left. \frac{1}{4}x^4 - \frac{1}{3}x^3 \right|_{-1}^1 \\ &= \left( \frac{1}{4} - \frac{1}{3} \right) - \left( \frac{1}{4}(-1)^4 - \frac{1}{3}(-1)^4 \right) \\ &= -\frac{1}{12} - \left( \frac{1}{4} + \frac{1}{3} \right) \\ &= -\frac{1}{12} - \frac{7}{12} \\ &= -\frac{8}{12} = \boxed{-\frac{2}{3}}. \end{aligned}$$

*Method 2:*

$$\begin{aligned}
 & \int_{-1}^1 x^3 - x^2 dx \\
 &= \left. \frac{1}{4}x^4 - \frac{1}{3}x^3 \right|_{-1}^1 \\
 &= \frac{1}{4}(1 - (-1)^4) - \frac{1}{3}(1 - (-1)^3) \quad \triangleleft \text{Homogen. \& Additive} \\
 &= \boxed{\frac{2}{3}}.
 \end{aligned}$$

*Example Notes:* In the second method, I evaluated *each term* between the limits—enabling me to combine similar expressions immediately. It also allowed me to factor out the constants  $1/4$  and  $1/3$  that I wasn't able to do in *Method 1*. Overall, in this problem at least, using the properties of the evaluation rule made the numerical calculations simpler, cleaner, and less prone to error.

You'll have to judge for yourself. At least try using these tricks yourself. ■



**9.3.** We are integrating

$$\int_1^2 x^{-3} dx,$$

the simple power rule will suffice.

$$\begin{aligned} \int_1^2 x^{-3} dx &= \left. \frac{x^{-2}}{-2} \right|_1^2 \\ &= -\frac{1}{2} x^{-2} \Big|_1^2 &< \text{Homogen. of EBLO} \\ &= \frac{1}{2} x^{-2} \Big|_2^1 &< \text{Reverse of EBLO} \\ &= \frac{1}{2} (1 - 2^{-2}) \\ &= \frac{1}{2} \cdot \frac{3}{4} = \boxed{\frac{3}{8}} \end{aligned}$$

Where, we have reversed the order of evaluation mostly in order to get rid of the negative sign at the beginning of the expression. The

integral, I knew, would be evaluating to a positive number, the negative sign is a bit out of place — so I got rid of it by reversing the order of evaluation.

This little trick can be used to avoid negative sign that have the potential of causing the neophyte algebra student to error — we don't want that to happen!

Example 9.3. ■

**9.4.** First we must decide how to solve the integral

$$\int_0^1 x\sqrt{x^2 + 3} dx.$$

Using my **Butterfly Method** we first look at the first formula on our list for formulas: **The Power Rule**. We can solve the problem by first solving the corresponding *indefinite integral*:

$$\begin{aligned} & \int x\sqrt{x^2 + 3} dx \\ &= \frac{1}{2} \int (x^2 + 3)^{1/2} 2x dx &< \begin{cases} u = x^2 + 3 \\ du = 2x dx \end{cases} \\ &= \frac{1}{2} \int u^{1/2} du &< \text{substitution} \\ &= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C \\ &= \frac{1}{3} (x^2 + 3)^{3/2} + C &< \text{resubstitute} \end{aligned}$$

Now, having used standard techniques to compute the *indefinite integral*, we can then evaluate the *definite integral*:

$$\begin{aligned}\int_0^1 x\sqrt{x^2+3} dx &= \frac{1}{3} (x^2+3)^{3/2} \Big|_0^1 &< \text{EBLO Trick} \\ &= \frac{1}{3} (4^{3/2} - 3^{3/2}) \\ &= \boxed{\frac{1}{3} (8 - 3\sqrt{3})}.\end{aligned}$$

This is one style for solving integrals that involve an *auxiliary substitution* to solve. An alternate procedure is discussed next.

Example 9.4. ■

**9.5.** You *should* have let  $u = 2x$ ,  $du = 2 dx$ , and used the trig formula (2) to obtain

$$\begin{aligned}\int_0^{\pi/2} \sin(2x) dx &= \frac{1}{2} \int_0^{\pi/2} \sin(2x) 2 dx \\ &= -\frac{1}{2} \cos(2x) \Big|_0^{\pi/2} \\ &= -\frac{1}{2} (\cos(\pi) - \cos(0)) \quad \triangleleft \text{EBLO Tricks} \\ &= -\frac{1}{2} (-1 - 1) \\ &= 1.\end{aligned}$$

Here, I have combined the calculation of the indefinite integral into the same line of development as the definite integral. I set up the substitution, but did not actually substitute; had I done so, I would have had a new variable of integration  $u$ , but my limits of integration are for the  $x$  variable — that's no good. Example 9.5. ■



**9.6.** This is the example seen earlier. We check whether we get the same answer. The integral is

$$\int_0^1 x \sqrt{x^2 + 3} dx$$

We apply substitution: Let  $u = x^2 + 3$ ,  $du = 2x dx$ , then

$$\int_0^1 x \sqrt{x^2 + 3} dx = \frac{1}{2} \int_0^1 (x^2 + 3)^{1/2} 2x dx$$

Rearrange the integrand and prepare it for the power rule, with the designated  $u$  and the calculated value of  $du$ , we inserted our **fudge factor**. We are ready for our substitution.

$$\begin{aligned} \int_0^1 x \sqrt{x^2 + 3} dx &= \frac{1}{2} \int_0^1 (x^2 + 3)^{1/2} 2x dx \\ &= \frac{1}{2} \int_3^4 u^{1/2} du \end{aligned} \quad \triangleleft \text{ substitute!}$$

Now where did the *new limits of integration* come from? They come from the **substitution formula**, of course. There is the *reasoning* for computing these limits:

*Calculation of New Lower Limit:* The substitution is  $u = x^2 + 3$  (N.B.: this is the  $u=g(x)$  function.) We say, when  $x$  is at its lower limit, what is the value of  $u$ ? The lower limit of  $x$  is  $x = 0$ . When  $x = 0$ , that is the corresponding value of  $u$ ? It would be  $u = 0^2 + 3 = 3$ . (N.B.: this is  $g(a)$ .) Or, more simply,

$$x = 0 \text{ and } u = x^2 + 3 \implies u = 3.$$

This is the *new lower limit*.

*Calculation of the New Upper Limit:* When  $x$  is at its upper limit, what is the corresponding value of  $u$ ?

$$x = 1 \text{ and } u = x^2 + 3 \implies u = 4.$$

This is the *new upper limit*.

Let's finish up our calculation and call it quits.

$$\begin{aligned}\int_0^1 x\sqrt{x^2+3} dx &= \frac{1}{2} \int_3^4 u^{1/2} du \\ &= \frac{1}{2} \frac{2}{3} u^{3/2} \Big|_3^4 \\ &= \frac{1}{3} (4^{3/2} - 3^{3/2}) \\ &= \frac{1}{3} (8 - 3\sqrt{3})\end{aligned}$$

The solution was rather long, but actually not. Just the last set of equations in the above display represent the entire solution to this definite integral.

Example 9.6. ■

**9.7.** This integral,

$$\int_{-2}^3 (1 - 4x)^{-5} dx$$

can be solved by the **Power Rule**. For the power rule formula, the  $u$  is the formula is the base of the power function. This forces us to say:  
Let

$$u = 1 - 4x$$

and,

$$du = -4 dx$$

*Change of Limit Calculations:*

$$\text{Lower Limit: } x = -2 \text{ and } u = 1 - 4x \implies u = 9$$

$$\text{Upper Limit: } x = 3 \text{ and } u = 1 - 4x \implies u = -11$$

*Calculation of the Integral:*

$$\begin{aligned}\int_{-2}^3 (1 - 4x)^{-5} dx &= -\frac{1}{4} \int_{-2}^3 (1 - 4x)^{-5} (-4) dx \\ &= -\frac{1}{4} \int_9^{-11} u^{-5} du \\ &= -\frac{1}{4} \frac{u^{-4}}{-4} \Big|_9^{-11} \\ &= -\frac{1}{16} u^{-4} \Big|_9^{-11} \\ &= -\frac{1}{16} ((-11)^{-4} - 9^{-4}) \qquad \text{(S-7)} \\ &= \frac{1}{16} \left( \frac{1}{11^4} - \frac{1}{9^4} \right).\end{aligned}$$

The second factor in line (S-7) is *negative*, standard algebraic methods dictate to reverse the order of subtraction, and pre-fixing a negative

sign; that is exactly what I did in the next line — the negative combined with a negative sign already there, to obtain a positive sign, which, of course, is not written. [Example 9.7.](#) ■

**9.8.** Here's what your eyes and mind should observe:

$$\int_{-2}^2 x^2 dx.$$

*Observation 1:* We are integrating over a *symmetric interval*.

*Observation 2:* The function  $x^2$  is even:  $(-x)^2 = x^2$ .

Let's solve the integral two ways. Not using symmetry, and then taking advantage of symmetry.

*Don't Use the Symmetry:*

$$\begin{aligned}\int_{-2}^2 x^2 dx &= \left. \frac{1}{3}x^3 \right|_{-2}^2 \\ &= \frac{1}{3}(2^3 - (-2)^3) \\ &= \frac{1}{3}(8 - (-8)) && \text{(S-8)} \\ &= \frac{1}{3}(8 + 8) \\ &= \boxed{\frac{16}{3}}.\end{aligned}$$

Notice that there was some tricky evaluations in line (S-8), not difficult, but for many people, this kind of “double negative” evaluation causes difficulties, confusion, and disorientation.



Use *Symmetry*:

$$\begin{aligned}\int_{-2}^2 x^2 dx &= 2 \int_0^2 x^2 dx &< (10) \\ &= 2 \left. \frac{1}{3} x^3 \right|_0^2 \\ &= 2 \frac{1}{3} (8 - 0) && \text{(S-9)} \\ &= \boxed{\frac{16}{3}}.\end{aligned}$$

The tricky evaluation of the first solution is gone. The line (S-9) now is very simple to evaluate; the chance of making a computation error is reduced quite a bit.

Example 9.8. ■

**9.9.** We are integrating an odd function over a symmetric interval of integration; therefore, by **Theorem 9.2**, we relatively almost instantaneously deduce that

$$\int_{-1}^1 x^3 dx = 0$$

Example 9.9. ■

**9.10.** Wow! The given integral is

$$\int_{-100}^{100} \frac{\sin(x)}{\sqrt{x^8 + x^4 + x^2 + 1}} dx.$$

This is nothing more than the integral of an odd function over a symmetric interval. Therefore,

$$\int_{-100}^{100} \frac{\sin(x)}{\sqrt{x^8 + x^4 + x^2 + 1}} dx = 0.$$

That was simple.

*Example Notes:* The “difficult part” of this problem is arguing that the integrand is odd. No problem. The numerator is  $\sin(x)$ , a notorious odd function. The denominator, is an even function. It is well-known that the ratio of an odd function and an even function results in an *odd function!* That’s all. ■

Example 9.10. ■

**9.11. Background Thinking.** The integrand  $f(x) = x^6 - x^3$  is an **even function** *minus* an **odd function**. We are integrating over a symmetric interval.

*Evaluation:*

$$\begin{aligned}\int_{-1}^1 x^6 - x^3 dx &= \int_{-1}^1 x^6 dx - \int_{-1}^1 x^3 dx \\ &= 2 \int_0^1 x^6 dx - 0 \\ &= 2 \frac{1}{7} \\ &= \frac{2}{7}\end{aligned}$$

*Example Notes:* Here, we separated the even function from the odd function using a *technique*, then utilized any symmetric properties on each integral. ■

Example 9.11. ■

# Important Points

## Outline of the genesis of area.

When first you met area it was, perhaps, when you were studying rectangles. Then, whether you were aware of it or not, *area of a rectangle* having base  $b$  and height  $h$  was *defined* to be

$$A_{\text{rect}} = bh.$$

**Intuitive Notion of Area.** Let's make a distinction between area that has been defined for a particular geometric shape, the rectangle, and the intuitive notion of area. The intuitive notion of area was that all geometric shapes have this area quantity, and that area had certain properties—properties consistent with the intuitive notion of area.

*Properties of Area.* In order to be consistent with the intuitive notion of area, the defined area must have certain properties.

- (Nonnegativity of Area.) Area is a nonnegative quantity.
- (Additivity of Area) If we have a region in the plane (i.e. some geometric shape) having area  $A$ , and we subdivide this region into two subregions, each having respective areas  $A_1$  and  $A_2$ , then  $A = A_1 + A_2$ .

## Important Points (continued)

- (Monotonicity of Areas.) Suppose we have two regions in the plane,  $R_1$  and  $R_2$ , such that  $R_1 \subseteq R_2$ . Then the area of  $R_1$  is less than or equal to the area of  $R_2$ .
- (Congruency and Area.) Congruent regions have equal area.

Armed with the definition of area for a rectangle and the intuitive properties of area, it is possible to deduce the notion of area for a great variety of geometric shapes: triangles, parallelograms, trapezoids, and so on.

The circle was another two-dimensional shape that has an area. From Euclid we know that the area of a circle of radius  $r$  is given by

$$A = \pi r^2 \quad \text{Area of a circle.}$$

By seeing these common shapes, calculating their area by their formulas, all the while keeping in mind the properties of area, we come to “know” the notion of area. However, that doesn’t change the fact that the notion of area is still an undefined quantity for a large number of shapes.

Important Point ■

**Define/Calculate** the area under the graph of  $f$ .

Because we are attempting to extend the notion of area to regions of (somewhat) arbitrary shape, our task is actually to *define* what we mean by area. This definition must be consistent with previous notions of area however. For example, if the region under the graph of  $f$  happens to be a rectangle or triangle, our new definition of area should reduce to that of the area of a rectangle or triangle.

Defining area in a new situation is nice, but we also want to develop methods of *calculating* this area.

For these reasons, the phrase ‘define/calculate’ is used.

Important Point ■



**Definition.** Assume the notation of the **constructive** definition of the definite integral. Let  $L$  be a number. We say that

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = L$$

provided it is true that for any  $\epsilon > 0$ , there exists a number  $\delta > 0$  such that

$$\left| \sum_{i=1}^n f(x_i^*) \Delta x_i - L \right| < \epsilon$$

whenever  $P$  is a partition of the interval  $[a, b]$  whose norm  $\|P\| < \delta$  and any choice of intermediate points  $x_i^*$ .

*Definition Notes:* The above is remarkably similar to **Definition 8.1**. The twist here is that this definition reflects the need to have the limit,  $L$ , independent of the choices for the intermediate points  $x_i^*$ .

■ Consequently, this limit is a true limit in that it has all the limit properties studied in the article on **Limits**. ■

Important Point ■

**Algorithm for expanding the symbol**  $\sum_{i=1}^n a_i$ .

**Problem:** Expand  $\sum_{i=1}^n a_i$ .

1. Put  $i$  equal to its initial value:  $i = 1$ . Evaluate the expression  $a_i$  for  $i = 1$  to obtain  $a_1$ . Include this term in sum: write

$$\sum_{i=1}^n a_i = a_1 +$$

2. Increment the value of the index to  $i = 2$ . Evaluate the expression  $a_i$  for  $i = 2$  to obtain  $a_2$ . Include this term in the sum: write

$$\sum_{i=1}^n a_i = a_1 + a_2 +$$

3. Increment the value of the index to  $i = 3$ . Evaluate the expression  $a_i$  for  $i = 3$  to obtain  $a_3$ . Include this term in the sum: write

## Important Points (continued)

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 +$$

4. Increment the value of the index to  $i = 4$ . Evaluate the expression  $a_i$  for  $i = 4$  to obtain  $a_4$ . Include this term in the sum: write

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 +$$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

- n. Increment the value of the index to  $i = n$ . Evaluate the expression  $a_i$  for  $i = n$  to obtain  $a_n$ . Add this term into the sum: write

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

5. Finished.

Important Point ■

**Some Common Summation Formulas.** Here are some famous summation formula that we will be using in this section.

$$1. \sum_{i=1}^n c = c + c + c + c + \cdots + c = nc$$

$$2. \sum_{i=1}^n i = 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

$$3. \sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$4. \sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

**Verbalizations.** Formula (1) can be read as the sum of the same number,  $c$ ,  $n$  times. Formula (2) is the sum of the first  $n$  integers. Formula (3) is the sum of the squares of the first  $n$  integers. And formula (4) is the sum of the cubes of the first  $n$  integers. Important Point ■

**Technical Definition.**

**Definition.** Assume the notation of the **constructive** definition of the definite integral. Let  $L$  be a number. We say that

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^n f(x_i^*) \Delta x = L$$

provided it is true that for any  $\epsilon > 0$ , there exists a number  $M > 0$  such that whenever the interval  $[a, b]$  is partitioned into  $n$  subintervals,  $n \geq M$ , of equal length  $\Delta x = (b - a)/n$ , it must be true that

$$\left| \sum_{i=1}^n f(x_i^*) \Delta x - L \right| < \epsilon$$

no matter the choice of the intermediate points  $x_i^*$ .

*Definition Notes:* Roughly speaking,  $L$  is the limit (hence the value of the definite integral) if  $L$  can be approximated by Riemann sums to any preselected degree of accuracy (that's the  $\epsilon$ -value) by partitioning up the interval  $[a, b]$  into a large enough number of intervals

## Important Points (continued)

(that's the role of the  $M$ ). The precision of the approximation does not depend on the intermediate points  $x_i^*$ , rather it depends only on the whether you have subdivided the interval into a sufficient number of subintervals (that's the  $M$  again). (That should make it clear!)

- This limit formulation makes it much easier to compute an integral. ■

Important Point ■

## Important Points (continued)

Have you ever cut out two paper strips of *exactly* the same width?

Actually, in some developments seen in *Calculus II* and *Calculus III*, it is inconvenient to require the subintervals to be of equal length—and actually makes it more difficult to develop the application.

Important Point ■

## Important Points (continued)

There is more area *above* the  $x$ -axis than *below* the  $x$ -axis.

Important Point ■



## Important Points (continued)

There is more area *below* the  $x$ -axis than *above* the  $x$ -axis.

Important Point ■

## Important Points (continued)

There is *equal* amounts of area below the  $x$ -axis as above the  $x$ -axis.

Important Point ■

### **Solution to in-line quiz.**

The *distributive law* states, for any  $a, b, c \in \mathbb{R}$ ,

$$a(b + c) = ab + ac.$$

Using this formula from left to right, we say that we distribute the  $a$  through the sum (or, multiply through by  $a$ ). Using the formula from right to left, we often say that we factor out the common factor of  $a$ .

In the step above, we have essentially used the distributive law from right to left. We have factored out the common factor of  $c$ , in the notation of the problem.

[Important Point](#) ■

**Solution to in-line quiz.**

The *associative law* states, for any  $a, b, c \in \mathbb{R}$ ,

$$(a + b) + c = a + (b + c).$$

This relation is important because we can only add two numbers at a time. In the case we want to add three numbers together, we can do so by adding the first two together, then add that result to the third. The associative law then states that sequence of operations yields the same result as taking the first number, and adding it to the sum of the second two numbers.

The *commutative law* states that for any  $a, b \in \mathbb{R}$ ,

$$a + b = b + a.$$

This law states that we can add two numbers together in any order.

## Important Points (continued)

In the above problem we have used a combination of the associative law and commutative laws to justify the equality. Here are some detail for the simple case of  $n = 2$ .

$$\begin{aligned}(a_1 + b_1) + (a_2 + b_2) &= a_1 + (b_1 + (a_2 + b_2)) &< \text{associative law} \\ &= a_1 + ((b_1 + a_2) + b_2) &< \text{associative law} \\ &= a_1 + ((a_2 + b_1) + b_2) &< \text{commutative law} \\ &= a_1 + (a_2 + (b_1 + b_2)) &< \text{associative law} \\ &= (a_1 + a_2) + (b_1 + b_2) &< \text{associative law}\end{aligned}$$

Now you see why it did the case of  $n = 2$ . That's more microminature algebraic than you've see in a long time. However, such "low level" knowledge of our arithmetic system gives us a better overall understanding of our mathematical universe.  $\mathfrak{D}\mathfrak{S}$  Important Point ■

## Important Points (continued)

We proceed as follows:

$$\begin{aligned}\sum_{k=5}^{10} k^3 &= \sum_{k=1}^{10} k^3 - \sum_{k=1}^4 k^3 && \triangleleft \text{from (18)} \\ &= \left[ \frac{10(11)}{2} \right]^2 - \left[ \frac{4(5)}{2} \right]^2 && \triangleleft \text{Sum Formula (4)} \\ &= [5(11)]^2 - [2(5)]^2 \\ &= \boxed{2,925}\end{aligned}$$

Important Point ■

## Important Points (continued)

This will work:  $\sum_{i=3}^{n+3} 2^{i-3}$ . Expand it out to verify.

Important Point ■

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