THE UNIVERSITY OF AKRON
Mathematics and Computer Science

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calculus menu

Article: Applications to Definite Integration

Directory

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Applications to Definite Integration

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1. Introduction

Prerequisite: Limits, Continuity, Differentiation, Integration.

In this article we present several applications to the definite integral that will illustrate how it is used to solve applied problems. These applications are just the "tip of the iceberg." There are many, many varied uses of the definite integral. As you enter into you own applied field of interest, no doubt you will stumble over several interesting applications.

When it comes to presenting applications, there is a small problem. The students who take *Calculus* have varied backgrounds: some have taken *Physics*, others have not; some have a scientific background, others have not.

Because of the different backgrounds of the students, it is difficult to present very applied illustrations of integration. The really interesting applications of the definite integral require deeper understanding of the applied field. For some applications it might take several weeks (months) to acquire the necessary background to understand the problem. Therefore, the kinds of applications that we look at are of two types:

- **Problems in areas were students have a common background.** All students taking *Calculus* have a common background *geometry*. Concepts such as *area*, *volumne*, *arclength*, and *surface area* would be familiar to every student. Problems of computing these quantities are very intuitive and geometric; hence, easy to understand.
- **"Quick-in, Quick-out" Applications.** There is a need to demonstrate some application to the student beyond geometry. Because of the nature of the students' background (varied) these applications must be "quick-in and quick-out." By that I mean, applications that require little time to explain the necessary background information before the analysis begins and simple enough that the analysis can be done in short order so we can exit the application after declaring victory. Traditional examples are problems involving the *center of mass* of a thin plate and the physical notion of *work*.

All these problems are analyzed in exactly the same way. As you see the analysis, try to abstract the process of analysis so that you can gain an understanding of the "standard" method of analyzing problems. At the end of this article, I will ask you write a paragraph or two that will describe the kinds of problems to which the definite integral can be applied. Watch for it.

2. The Area of a Region in the Plane

In the next section, we formally state the problem and develop the calculating formula. In Section 2.2 we illustrate the use of the formula by example.

2.1. Developing the Formula

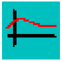


Figure 1

The Problem: Given two functions $y = f(x)$ and $y = g(x)$ defined over an interval closed interval $[\,a, b\,]$, *define/calculate* the area of the region, $\mathcal{R}$, bounded (or enclosed) by these two curves and the vertical lines $x = a$ and $x = b$. (See Figure 1.)

- *The Idea behind the Solution*: The solution to this problem follows the constructive nature of the definite integral. This reference describes the process in layman terms.

The Solution to the Problem. Let's begin by labeling our target region $\mathcal{R}$ for convenience. The problem is to define/calculate the area of $\mathcal{R}$, which will be labeled A.

Let $n \in \mathbb{N}$ be given. Create a partition P of the interval $[\,a, b\,]$ using partition points x_i. Out of the i^{th} interval, $[\,x_{i-1}, x_i\,]$, which has width $\Delta x_i = x_i - x_{i-1}$, choose an intermediate point x_i^*.

Figure 2 Draw vertical lines at the partition points x_i. These vertical lines subdivide the target region into vertical strips. Label these vertical strips as $\mathcal{R}_1$, $\mathcal{R}_2$, ... , R_n, and designate the areas of these regions by ΔA_1, ΔA_2, ... , ΔA_n, respectively. It is clear from Figure 2 that

$$A = \sum_{i=1}^{n} \Delta A_i, \tag{1}$$

meaning *the whole is the sum of its parts.*

Figure 3 For each i, construct a rectangle bounded by the vertical lines $x = x_{i-1}$ and $x = x_i$ and the horizontal lines $y = f(x_i^*)$ and $y = g(x_i^*)$. (See Figure 3.)

The area of the i^{th} rectangle approximates ΔA_i, the area of the i^{th} subregion, $\mathcal{R}_i$. Thus,

$$\Delta A_i \approx |f(x_i^*) - g(x_i^*)|\, \Delta x_i. \tag{2}$$

From (1) and (2) we then have,

$$A \approx \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\, \Delta x_i. \tag{3}$$

A hard look at (3) leads to the realization that the right-hand side is a Riemann sum of the function $L(x) = |f(x) - g(x)|$. Recalling Theorem 7.4, which states that the limit of Riemann sums is an integral provided the function in question is continuous, we obtain

$$\begin{aligned} A &= \lim_{||P||\to 0} \sum_{i=1}^{n} |f(x_i^*) - g(x_i^*)|\, \Delta x_i \\ &= \int_a^b |f(x) - g(x)|\, dx. \end{aligned}$$

Thus,

$$A = \int_a^b |f(x) - g(x)|\, dx. \tag{4}$$

This completes the development of the solution. □

EXERCISE 2.1. Read the constructive solution to the problem, and read the *Idea behind the solution* and delineate the duality between each (i.e., the formal construction of the solution and the layman description of the solution).

Let's formalize formula (4) into a theorem.

Theorem 2.1. (Area between two Curves) *Let f and g be two functions defined and continuous over the closed interval $[\,a, b\,]$. Then the area bounded by the graphs of f and g is given by*

$$\boxed{A = \int_a^b |f(x) - g(x)|\, dx} \tag{5}$$

Theorem Notes: Equation (5) is a nice looking formula, but some experience is needed to learn how to use it.

▪ The first step toward using the area formula (5) is to remove the absolute values. The main tool for doing this is the additive property of limits for definite integrals.

▪ The letters used to describe the functions, f and g, were arbitrary (not really). The real functions may have most any name: $H(x)$, $\phi(x)$, $w(x)$, etc.

▪ The variable of integration, x, is a dummy variable. Perhaps the original function f and g are functions of t instead of x. Just replace the letter x with the letter t. ▪

In Section 2.2 an "easy" method is described to help you analyze area problems. Let's content ourselves for now by having some very simple examples to illustrate the basic area formula (5).

EXAMPLE 2.1. Consider the two functions $f(x) = x$ and $g(x) = x^2$, for $0 \le x \le 1$. Find the area bounded by there two functions.

Try a simple problem for yourself.

EXERCISE 2.2. Define functions $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 + 1$ over the interval $[0, 2]$. Find the area of the region bounded by the graphs of f and g. (The first step is to make a sketch of the region. Follow the solution method of Example 2.1.)

2.2. The Method of Rules

In the previous section the basic calculating formula for area, equation (5), was developed. How can we develop and organize our knowledge of this formula in order to solve area problems? In this section, we introduce a method of analyzing area problems called the **Method of Rules**.

The method, if properly understood and applied, will help you to analyze and solve even quite complex area problems. This method is a systematic way of applying the basic area formula (5). In a larger sense, the **Mehod of Rules** illustrates how mathematicians can take a basic formula, such as equation (5), and stretch and twist it around to apply it in diverse settings.

Upon finishing the study of the **Method of Rule**, the reader must surely wonder why so much time and energy was spent explaining and illustrating this method. Even though the **Method of Rules** is advertised as a method of solving area problems, the same basic techniques can be used to set up *limits of integration* for double and triple integrals. This is one of the big problems third semester students of *Calculus* encounter; the **Method of Rules** can be adapted to help students master the delicate art of setting up limits of integration for multiple integrals.

Take a look at the basic formula:

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

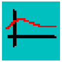


Figure 4

How can we interpret the integrand, $|f(x) - g(x)|$? For any given x, $a \leq x \leq b$, $|f(x) - g(x)|$ is the length of the (cross-sectional) line segment that extends from the *lower boundary curve* of the region $\mathcal{R}$ to the *upper boundary curve*. This observation is key to the **Method of Rules**. Before doing any examples, if you will bear with me, I'll introduce some terminology and methodology.

Let's begin by restating the problem under consideration in greater generality.

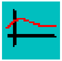


Figure 5

The Problem. Suppose we are given a region, $\mathcal{R}$, in the Cartesian plane. The problem is to *define/calculate* the *area* of the region $\mathcal{R}$.

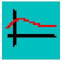


Figure 6

Terminology. The intersection of $\mathcal{R}$ with a straight line shall be called a *rule* of $\mathcal{R}$ (or, simply a rule, if the region is understood). Rules may be horizontal, vertical, or neither.

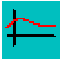


Figure 7

▪ *Generating Rules.* Let ℓ be a number line and let the numbers on the ℓ-axis are symbolically referenced by the letter s. Choose a number s on the ℓ-axis. The line ℓ and the choice of the number s generate, in a natural way, a rule of $\mathcal{R}$. We obtain this rule by intersecting $\mathcal{R}$ with the line that's perpendicular to ℓ and passes through s. Call this rule the *rule generated* by s (and the ℓ-axis).

We only consider regions $\mathcal{R}$ that satisfy certain conditions. These conditions are described in the form of ...

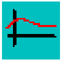


Figure 8

Assumptions. Let $\mathcal{R}$ be a region in the plane. Suppose we can find a number line, say the ℓ-axis, having the property that given any number s on this line, the rule (of $\mathcal{R}$) generated by s is either empty or is a line segment of finite length. Define

$$L(s) = \text{length of the rule generated by } s.$$

This defines a function L, called a *rule function of* $\mathcal{R}$. We further assume the function L is a continuous function of s.

Assumption Notes: The line ℓ can be the x-axis, the y-axis, or *any* of the other famous axes as long as they satisfy the above assumptions.

▪ For a given region $\mathcal{R}$, there may be several axes that satisfy the assumptions. This is often exploited as a technique of integration.

▪ If the rule generated by s is empty, then the rule misses the region $\mathcal{R}$; in this case, $L(s) = 0$.

▪ It is worthwhile to emphasize that L is a *function of position* on the number line ℓ which is usually the *axis of abscissas* (the horizontal axis) or the *axis of ordinates* (the vertical axis).

▪ It may be more convenient to think of $L(s)$ as the cross-sectional width of the region at s. As we will see, if the cross-sectional widths of a region are known, then the area of that region is known—the area of a region is determined by its cross-sections.

- The requirement that the rule must be a line segment can be relaxed. The rule can be the finite union of disjoint line segments.
- The *continuity* of the *rule function*, L, is not really required; we do require that L be (*Riemann*) *integrable*. ■

Statement of the Solution. Let $\mathcal{R}$ be a region in the plane and suppose there is a line ℓ that satisfies the **Assumptions** above. For any s on the line ℓ, define

$$L(s) = \text{length of the rule generated by } s.$$

Finally, let a be the left-most extremity of $\mathcal{R}$, and b be the right-most extremity of $\mathcal{R}$. (See Figure 8.) Then the area, A, of the region $\mathcal{R}$ is given by

$$\boxed{A = \int_a^b L(s)\, ds} \tag{6}$$

Important Point. Note that the variable of integration is s, the variable that symbolizes a number on the line ℓ.

Where did this formula come from? It's an obvious abstraction of the basic area formula (5). Indeed, suppose $\mathcal{R}$ is a region described in Theorem 2.1: $\mathcal{R}$ is bounded by the two functions f and g over the interval $[\,a, b\,]$. (See Figure 1.) Take the axis ℓ to be the x-axis. Choose any x, $a \le x \le b$, and draw the vertical rule generated by x. The length of this vertical rule, $L(x)$ is given by

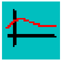


Figure 9

$$L(x) = \underbrace{|f(x) - g(x)|.}_{\text{Why?}} \tag{7}$$

Having made these observations, substitute (7) into the basic area formula (5), to obtain,

$$A = \int_a^b L(x)\,dx,$$

where L is the (vertical) rule for the region $\mathcal{R}$. This formula is obviously a version of (6).

We finally begin a set of examples and exercises to illustrate the use of the basic area formula (5), but, more importantly, to illustrate the method of calculating areas using equation (6).

Let's begin by revisiting Example 2.1, but this time we'll use the **Method of Rules** to solve the problem.

Example 2.2. (Vertical Rules) Consider the two functions $f(x) = x$ and $g(x) = x^2$, for $0 \leq x \leq 1$. Find the area bounded by there two functions.

Now you rework Exercise 2.2 using the **Method of Rules**. Follow the solution methods of Example 2.2.

Exercise 2.3. (Vertical Rules) Define functions $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 + 1$ over the interval $[\,0, 2\,]$. Find the area of the region bounded by the graphs of f and g.

In the previous example and exercise, the limits of integration were pretty much given. In this example, the limits of integration must be found.

Example 2.3. (Vertical Rules) Find the area of the region, $\mathcal{R}$, that is bounded by the curves $y = x^2 - 2x - 3$ and $y = 3x - 7$.

Let's not forget that the **Method of Rules** is not tied down to one axis. To demonstrate the versatility of the *Method of Rules*, let's solve exactly the same problem, but with *horizontal rules* instead of *vertical rules*.

Example 2.4. (Horizontal Rules) Find the area of the region, $\mathcal{R}$, that is bounded by the curves $y = x^2 - 2x - 3$ and $y = 3x - 7$.

These two examples illustrate the simplest situation; namely, a region bounded by two functions. Despite the simplicity, they do demonstrate the basic pattern of thinking that goes into solving area problems.

We have seen from these examples the need to be able compute the length of vertical or horizontal rules. The following rules are self-obvious, but I will state them for easy reference.

Calculation of Rules: Rules. Suppose you have a rule (a line segment) whose endpoints are $(\,x_1, y_1\,)$ and $(\,x_2, y_2\,)$.

1. (Vertical Rules) Suppose the rule is vertical (this implies $x_1 = x_2$), then the length of the rule is given by

$$\text{length} = |y_1 - y_2|. \tag{8}$$

This formula can be realized as the "y-coordinate (ordinate) of the *upper-most point* minus the y-coordinate (ordinate) of the *lower-most* point." (It was this rule that was used in the solution to EXAMPLE 2.3. See the calculation preceding equation (S-5).)

2. (Horizontal Rules) Suppose the rule is horizontal (this implies $y_1 = y_2$), then the length of the rule is given by

$$\text{length} = |x_1 - x_2|. \tag{9}$$

This formula can be realized as the "x-coordinate (abscissa) of the *right-most* point minus the x-coordinate (abscissa) of the *left-most* point." (This rule was used for the calculation of the rule length in EXAMPLE 2.4. See the calculation preceding equation (S-8).)

In the above stated rules, x is the variable that corresponds to the axis of abscissas (the horizontal axis) and y corresponds to the axis of ordinates (the vertical axis); naturally, when the two axes are labeled differently, the student should be able to translate these rules.

EXERCISE 2.4. Suppose we have a horizontal rule in the st-plane. Label the endpoints appropriately and write down a formula for the length of this rule.

In general, how do you compute the *limits of integration*? In previous examples I have illustrated the reasoning; in previous exercises you have determined the limits of integration yourself (I hope). The following two shadow boxes show the way.

How to find the Limits of Integration.

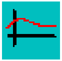


Figure 10

Vertical Rules. The figure to the left gives a picture of how to find the limits of integration when using vertical rules. The lower limit is located by finding the left-most extremity of the region. The upper limit of integration is found by locating the right-most extremity of the region.

How to find the Limits of Integration.

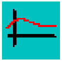


Figure 11

Horizontal Rules. The figure to the left gives a picture of how to find the limits of integration when using vertical rules. The lower limit is located by finding the lower-most extremity of the region. The upper limit of integration is found by locating the upper-most extremity of the region.

Limits of Integration Notes: These "extremities" of a region can be found in many different ways: graphically, algebraically, or using differential calculus. Not all methods work in any given situation.

▪ These "extremities" can occur at the intersection of two explicitly defined curves. The intersection points can be identified using graphical methods or by solving equations (if possible). Many of the examples and exercises in this tutorial will be of this type.

▪ Sometimes you must use differential calculus to find the limits of integration. Look at Figure 11, the upper and lower horizontal lines you see there are set at the extremities of the region. These lines are horizontal tangents to the boundary curve. You can well imagine the need to use differentiation to locate the exact positions of these lines. ▪

The rules on rules and the method of finding the limits of integration will aid you in the successful completion of the following two exercises. Draw a picture of the region, and following the chain of thought of Examples 2.3 and 2.4, complete the solutions without error.

Exercise 2.5. (Vertical Rules) Find the area of the region, $\mathcal{R}$, that is bounded by the curves $y = 3x$, $y = 2x$, and $x = 3$.

Exercise 2.6. (Horizontal Rules.) Find the area of the region, $\mathcal{R}$, that is bounded by the curves $y = x^2$ and $y = \sqrt{x}$.

How do we go about making the decision to use vertical or horizontal rules? This is a complicated question to answer; however, the following rules gives you some basic guidelines for choosing.

Horizontal or Vertical Rules?
As a general rule, if the boundary curves of the region are expressible as functions of the abscissas (the x) then *vertical rules* are the natural choice. In this case, x will be the variable of integration. If the boundary curves are expressible as functions of the ordinates (the y) then *horizontal rules* are the natural choice; the ordinate (the y) will be the variable of integration.

You'll note that in Example 2.3 the boundary curves were written in terms of x and we did indeed used *vertical rules*, as is prescribed by the above recommendations. In Example 2.4, the boundary curves were written in terms of x, but we wanted to *force* the use of *horizontal rules*. Since we wanted to use horizontal rules, it is inferred from the above recommendation that the boundary curves should be written as a function of y—and that is what we did as part of the solution to Example 2.4.

Question. Suppose the problem is to calculate the area enclosed between the two curves $x = y^2$ and $x - y = 2$. When setting up the area integral, what would be the "natural choice" to use: horizontal or vertical rules?

(a) Vertical Rules (b) Horizontal Rules

• Question. In the above remark, why did I use the phrase, "the variable of integration will be on the axis of abscissas (the x-axis)," rather than saying something like, "the variable of integration will be the variable x"?

The above rules tell you the *natural choice* of the rule. There are situation for which you would choose the "unnatural choice."

- **Using Vertical Rules**

Let's look at more examples in a more organized manner. Here we shall solve area problems using vertical rules. But first, let us review when it is appropriate to use vertical rules.

> When to use Vertical Rules?
> Vertical Rules would be the *natural choice* when the boundary curves of the region are expressible as functions of the abscissa (i.e., the x variable). In this case, the variable of integration will be on the axis of abscissas (the x-axis).

If you are the kind of person who likes a framework for solving problems, then these rules are for you. Here is a sequence of steps one typically follows, either implicitly or explicitly, for working area problems

The Seven Step Method.

1. Draw the picture of the region.
2. Determine the kind of rule to be used (horizontal or vertical) and the variable of integration. Write down the form of the area integral.
3. Find the limits of integration. This is done by Determining the extremities of the region—this may involve finding the points of intersection of the boundary curves. Now update the area integral using your new limits of integration.
4. Find the rule function.
5. Set up the area integral
6. Solve the area integral.
7. Present the answer. ■

Next is a *Skill Level* 0 problem that will illustrate how to solve an area problem using the above framework.

EXAMPLE 2.5. (Skill Level 0) Consider the two curves $y_1 = 4 - x^2$ and $y_2 = 1 - 2x$. Find the area enclosed by these curves.

Now let's review how to find the limits of integration.

How do you find the Limits of Integration?

Assumption: We are using Vertical Rules. Draw the region.

- **The Lower Limit of Integration.** Find the *left-most extremity* of the region and draw a vertical rule there. Characteristic of this rule is that it will touch the region at only one point and none of the region lies to the left of it. This vertical rule corresponds to some number a on the horizontal axis; find the number a. The number a will be the *lower limit of integration.*
- **The Upper Limit of Integration.** Find the *right-most extremity* of the region and draw a vertical rule there. Characteristic of this rule is that it will touch the region at only one point and none of the region lies to the right of it. This vertical rule corresponds to some number b on the horizontal axis; find the number b. The number b will be the *upper limit of integration.*

Important Point. This was the reasoning used in Example 2.5. Typically, the left-most and right-most extremities of a region can be obtained in one of two situations: the functions have a restricted domain, the endpoints of the domain constitute the limits of integration (see Example 2.6 below), or the, in the case the domains of the boundary curves are unrestricted, the extremities occur when the boundary curves intersect.

Use Example 2.5, the point above on how to find the limits of integration to solve the following problem. Solve this problem using the seven step format.

Exercise 2.7. (Skill Level 0) Consider the two functions $f(x) = (x-2)^2 - 4$ and $g(x) = x$. Find the area bounded by these two curves.

In Example 2.5, the region was of such a nature that one of the two curves given was always above the other curve given. This is not always the case. The next level of complexity is the have a region enclosed by two curves that cross each other.

Example 2.6. (Skill Level 1) Consider the two functions $f(x) = x$ and $g(x) = x^2$, for $0 \leq x \leq 2$. Find the area bounded by there two functions.

Exercise 2.8. Find the area bounded by the two curves $f(x) = \sin(x)$ and $g(x) = \cos(x)$, for $0 \leq x \leq \pi/2$.

Exercise 2.9. *Set up* the integral that is necessary to calculate the area bounded by the two curves $H(x) = \dfrac{2}{1+x^2}$ and $K(x) = x^2$. *Do not solve.*

- **Using Horizontal Rules**

Now let's look at some examples using horizontal rules. Whether you use vertical rules or horizontal rules, the reasoning process is always the same. First recall when to use horizontal rules.

When to use Horizontal Rules?
Horizontal Rules will be the *natural choice* when the boundary curves of the region are expressible as functions of the ordinate (i.e., the y variable). In this case, the variable of integration will be on the axis of ordinates (the y-axis).

EXAMPLE 2.7. (Skill Level 0) Find the area bounded by the two curves $f(y) = y^3$ and $g(y) = y$ in the first quadrant.

The last example was solved using the seven step method enumerated earlier.

EXERCISE 2.10. The graphs of the two functions $f(y) = y^4$ and $g(y) = (y + 2)^2$ enclose a region in the xy-plane. Calculate the area of this region.

How do you find the Limits of Integration?

Assumption: We are using Horizontal Rules. Draw the region.

- **The Lower Limit of Integration.** Find the *lowermost extremity* of the region and draw a horizontal rule there. Characteristic of this rule is that it will touch the region at only one point and none of the region lies below it. This horizontal rule corresponds to some number a on the vertical axis; find this number a. The number a will be the *lower limit of integration.*
- **The Upper Limit of Integration.** Find the *uppermost extremity* of the region and draw a horizontal rule there. Characteristic of this rule is that it will touch the region at only one point and none of the region lies above it. This vertical rule corresponds to some number b on the vertical axis; find this number b. The number b will be the *upper limit of integration.*

In the next example, the two boundary curves interchange positions. The method analysis is exactly the same, but the rule function is now a piecewise defined function.

EXAMPLE 2.8. (Skill Level 1) Find the area bounded by the two curves $x = y^2$ and $x = y^3 - 4y + 4$.

EXERCISE 2.11. Consider the region bounded by the curves $x = y^2 - 6y$, $x = |y - 1|$. *Set up* the integral representing the area bounded by these two curves.

- **Set up Strategies**

In the previous paragraphs, some rules for determining the *natural* choice of rule orientation were given:

- If the boundary curves are written as functions of x (the abscissa), then *vertical rules* are the natural choice.
- If the boundary curves are written as functions of y (the ordinate), then *horizontal rules* are the natural choice.

In this section we look at several situations that extend the ideas already presented.

- *Changing Rules as a technique of integration.* In the above examples, we always made the "natural choice" of rule orientation. Sometimes the natural choice is not the best choice.

Suppose the natural choice is to use vertical rules. Imagine a problem in which you set up the area integral using the natural choice (say vertical rules), but the resultant integral is too difficult to solve; we still want to solve the area problem (exactly if possible), what do we do? Try setting up the area integral for a horizontal rules. Maybe we can get an integral we can solve.

$$\underbrace{A = \int L_v(x)\,dx}_{\text{Can we solve it?}} \text{ if not, } \underbrace{A = \int L_h(y)\,dy}_{\text{Can we solve it?}}$$

where, L_v is the vertical rule function and L_h is the horizontal rule function. (No limits of integration have been specified—I ran out of letters to use.)

Here is an example to illustrate this concept.

EXAMPLE 2.9. Calculate the area bounded by the curve given by $f(x) = \sqrt{1+\sqrt{x}}$, $0 \le x \le 4$, and the x-axis.

EXERCISE 2.12. Calculate the area enclosed by the curve given by $y = \sqrt[3]{1+\sqrt{x}}$, $0 \le x \le 49$ and the x-axis.

1. set up the area integral using vertical rules, and observe that this integral is unsolvable.
2. set up the area integral using horizontal rules, and observe that this integral is solvable. Solve it.

EXERCISE 2.13. Find the area enclosed by the graphs of the two curves $f(x) = (1+\sqrt{x})^{1/3}$ and $x - 49y + 49 = 0$. (*Note*: These two curves intersect at $(0, 1)$ and $(49, 2)$.)

1. set up the area integral using vertical rules, and observe that this integral is unsolvable.

2. set up the area integral using horizontal rules, and observe that this integral is solvable. Solve it.

■ *"Forcing" an unnatural choice of rules.* This is a useful exercise to re-enforce your understanding of the process of setting up an area integral.

In the next example, we solve the EXAMPLE 2.5 using *horizontal rules*. It is more natural to solve this problem as we did above, using vertical rules: The boundary curves are expressed as functions of x, so standard reasoning dictate the use of vertical rules.

EXAMPLE 2.10. (Force Horizontal Rules) Consider the two curves $y_1 = 4 - x^2$ and $y_2 = 1 - 2x$. Find the area enclosed by these curves.

EXERCISE 2.14. (Force Horizontal Rules) Consider graphs of the two functions $f(x) = 1/x^2$ and $g(x) = 8x$ restricted to the interval $0 \leq x \leq 2$. Using horizontal rules, find the area of the region between these two curves.

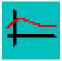


Figure 12

■ *Boundary Equations given as equations.* Sometimes the boundary curves can easily be described using equations rather than functions. In this case, there is no "natural choice" of rules—the ease of set up and the difficulty of the restultant integral are the factors that go into the choice of the orientation of the rule.

In the following two exercises, draw a picture of the region and decide what method is best for each situation. Then *set up* but *do* not evaluate the area integral.

Exercise 2.15. Consider the two curves $x^2 + y^2 = 2$ and $y = x^2$. Set up the area integral for the area of the region above the parabola and below the circle.

Exercise 2.16. Consider the two curves $x^2 + y^2 = 2$ and $y = x^2$. Set up the area integral for the area of the region bounded by the positive x-axis, the circle and the parabola.

3. The Volume of a Solid: The Method of Slicing

Another geometric application to the definite integral is the problem of defining and calculating the volume of a solid. We use the word 'defining' because, with respect to your mathematical history, only volumes of simple shapes have been defined—it is from these shapes that we inherit a basic understanding of what the word volume means.

In this, as well as the next section, we take a more mathematical approach to the problem of defining exactly what is meant by the volume of a solid. To that end, we must recall the definition and volume of the *right cylindrical solid*.

Construction of a Right Cylindrical Solid. Let $A > 0$ and $h > 0$ be numbers, and R a region in the xy-plane having area A. The union of all vertical line segments of length h having lower endpoint in the region R is called a *right cylindrical solid* of thickness (or height) h and cross-sectional area A. The region R is called the *base* of the solid.

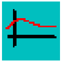
We will take as a basic fact that the *volume* of a cylindrical solid is given by

$$V = Ah. \tag{1}$$

A Pedestrian Description of a right cylindrical solid can also be given.

Of course, any rotation and translation of a right cylindrical solid is again a right cylindrical solid. In this way we obtain cylinders of different orientations and locations.

3.1. Developing the Formula



The Problem: Given a solid $\mathcal{S}$ in $\mathbb{R}^3$, *define/calculate* the volume of the solid, $\mathcal{S}$. (See Figure 1.)

Figure 1

Assumptions. We will assume that the solid $\mathcal{S}$ is of such a nature that whenever we intersect the solid with a plane perpendicular to the x-axis the cross-sectional area is known.

▪ *The Idea behind the Solution*: The method of solution is described here in a pedestrian sort of way.

The Solution to the Problem. Let's begin by labeling our target region $\mathcal{S}$ for convenience. The problem is to define/calculate the volume, which will be labeled V, of $\mathcal{S}$.

From the assumptions, when we intersect the solid $\mathcal{S}$ with a plane perpendicular to the x-axis, the cross-sectional area is known. This area will be a function of where we made the cross-section. Let $x \in \mathbb{R}$, intersect the solid $\mathcal{R}$ with a plane perpendicular to the x-axis at x and let $A(x)$ denote the cross-sectional area. Thus,

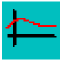


Figure 2

$$A(x) = \text{cross-sectional area of } \mathcal{S} \text{ at } x.$$

From Figure 2, there is a natural range of values for x, say $a \leq x \leq b$. Thus, $A(x) = 0$ for all x *not* in the interval $[\,a, b\,]$.

Let $n \in \mathbb{N}$ be given. Create a partition P of the interval $[\,a, b\,]$ using partition points x_i. Out of the i^{th} interval, $[\,x_{i-1}, x_i\,]$, which has width $\Delta x_i = x_i - x_{i-1}$.

At each of the points x_i, $i = 0, 1, 2, 3, \ldots, n$, slice the solid with a plane at x_i. This slicing subdivides the solid $\mathcal{S}$ into n sections $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3, \ldots, \mathcal{S}_n$:

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \cdots \cup \mathcal{S}_n.$$

For each i, let ΔV_i be the volume of the solid $\mathcal{S}_i$. Naturally,

$$V = \Delta\mathcal{S}_1 + \Delta\mathcal{S}_2 + \Delta\mathcal{S}_3 + \cdots + \Delta\mathcal{S}_n$$

or,

$$V = \sum_{i=1}^{n} \Delta\mathcal{S}_i \tag{3}$$

For each i, we want to approximate the value of $\Delta\mathcal{S}_i$. The section $\mathcal{S}_i$ is that portion of $\mathcal{S}$ that lies between the vertical planes intersecting the x-axis at $x = x_{i-1}$ and $x = x_i$. When $\Delta x_i = x_i - x_{i-1}$ is real small, the section $\mathcal{S}_i$ looks very flat … almost like a thin cylindrical solid of thickness Δx_i. Choose an intermediate point x_i^* from the interrval $[\,x_{i-1}, x_i\,]$ and approximate the section $\Delta\mathcal{S}_i$ with a cylindrical solid of thickness Δx_i and cross-sectional area $A(x_i^*)$. Thus, from (1),

$$\Delta\mathcal{S}_i \approx A(x_i^*)\Delta x_i.$$

Now, substituting this approximation into (3), we obtain

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i. \tag{4}$$

You will note that the sum on the right is a Riemann Sum. Consequently, if the cross-sectional area function $A(x)$ is Riemann integrable), then

$$\lim_{||P||\to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\,dx$$

This observation combined with (4) leads us to make the following

...

Definition 3.1. Let $\mathcal{S}$ be a solid and let $A(x)$ be the area of the cross-section obtained by intersecting $\mathcal{S}$ by a plane perpendicular to the x-axis at the number x, $a \leq x \leq b$. The volume of the solid $\mathcal{S}$ is defined to be

$$V = \int_a^b A(x)\,dx, \tag{5}$$

provided the integral exists (i.e., provided $A(x)$ is Riemann integrable).

Definition Notes: It is sometimes difficult at first to determine the values of the limits of integraton, a and b. The lower limit of integration, a, is the *largest* number such that $A(x) = 0$, for all $x < a$; similarly, the lower limit of integration, b, is the *smallest* number such that $A(x) = 0$, for all $x > b$

■ There is nothing particularly sacred about the x-axis. Even though we have developed the area volume formula (5) by slicing perpendicular to the x-axis, we could have very well sliced perpendicular to the y-axis—or perpendicular to *any* axis.

■ The guiding principle is this: Is there an axis such that any cross-section of $\mathcal{S}$ taken perpendicular to that axis has an area that can be computed? ■

3.2. Applying the Formula

We begin illustrating the use of the basic volume formula (5) by looking at solids obtained by rotation. Rotational solids are frequently seen in your daily life. Any solid that have been turned on a lathe is of this type.

After discussing how to compute the volume of a solid of revolution, we then look at solids that are not (necessarily) rotational, but have known cross-sections.

- **Rotation about axis parallel to x-axis**

Suppose we have a curve $y = f(x)$, $a \le x \le b$, and we wish to rotate the curve about the x-axis. **Question:** What is the volume of the resultant solid of revolution?

If we slice the solid perpendicular to the x-axis at the number x we see that the cross-section of the solid is a circular disk. The radius of this circular disk is $|f(x)|$. Consequently, the cross-sectional area at x of the solid is given by $A(x) = \pi[|f(x)|]^2$, or, more simply,

$$A(x) = \pi[f(x)]^2, \quad a \leq x \leq b.$$

Thus, by the volume by slicing formula (5), we have

$$V = \int_a^b \pi[f(x)]^2 \, dx \tag{6}$$

This is just one of many different formulas that follow from the basic volume formula. The above formula may be memorized, but it really isn't necessary. It's just equation (5) and the formula for the area of a circle; that former you do need to memorize, the latter you already know.

Example 3.1. The curve $f(x) = x^2$, $1 \leq x \leq 2$, is rotated about the x-axis. Find the volume of the solid of revolution.

We can also calculate volumes of rotation around axes parallel to the x-axis.

EXAMPLE 3.2. Consider the curve $f(x) = 1 + 4x^3$, $0 \leq x \leq 2$. This curve is rotated about the line $y = -1$. Find the volume of the solid of revolution.

- **Rotation about axis parallel to y-axis**
- **Slicing of Non-Rotational Objects**

4. The Volume of a Solid: The Method of Shells

4.1. Developing the Formula

4.2. Applying the Formula

- **Rotation about axis parallel to x-axis**
- **Rotation about axis parallel to y-axis**

Solutions to Exercises

2.1. The partitioning of the interval $[a, b]$ corresponds to cutting the paper into strips of various widths. The i^{th} paper strip has width $\Delta x_i = x_i - x_{i-1}$.

The choice of the numbers x_i^* corresponds to deciding where to cut your paper strip to more precisely conform to the height of the region where you plan to lay that particular paper strip down. You cut your i^{th} paper strip so that its total height is $|f(x_i^*) - g(x_i^*)|$.

The *Riemann Sums* are just the total area of the paper strips.

Exercise 2.1. ■

2.2. From the basic area formula, equation (5), we have

$$A = \int_a^b |f(x) - g(x)|\, dx,$$

where, in your problem, $f(x) = \frac{1}{2}x - 1$ and $g(x) = x^2 + 1$, $a = 0$, and $b = 2$. Thus,

$$A = \int_0^2 |(\tfrac{1}{2}x - 1) - (x^2 + 1)|\, dx. \tag{A-1}$$

Draw a graph of the two functions. You can easily see the graph of g is always above the graph of f. This means $g(x) \geq f(x)$ over the interval $[\,0, 2\,]$ and implies that

$$|f(x) - g(x)| = g(x) - f(x) = (x^2 + 1) - (\tfrac{1}{2}x + 1).$$

Substituting this into (A-1) we obtain,

$$
\begin{aligned}
A &= \int_0^2 (x^2+1) - (\tfrac{1}{2}x+1)\,dx \\
&= \int_0^2 x^2 - \tfrac{1}{2}x\,dx && \triangleleft \text{ Combine first!} \\
&= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^2\Big|_0^2 && \triangleleft \text{ Power Rule} \\
&= \frac{8}{3} - 1 = \frac{5}{3} && \triangleleft \text{ Evaluate!}
\end{aligned}
$$

Presentation of Answer: $\boxed{A = \dfrac{5}{3}}$.

Exercise 2.2. ■

2.3. Again, you should have sketched the graph. Having done that, you can observe, graphically, that $\frac{1}{2}x - 1 < x^2 + 1$, for all $0 \le x \le 2$.

The Rule Function: For any given x, $0 \le x \le 2$,

$$\begin{aligned} L(x) &= |f(x) - g(x)| \\ &= g(x) - f(x) && \triangleleft \text{ since } g(x) < f(x) \qquad \text{(A-3)} \\ &= (x^2 + 1) - (\frac{1}{2}x - 1) && \triangleleft \text{ write in terms of } x \\ &= x^2 - \frac{1}{2}x + 2 && \triangleleft \text{ combine/simplify} \end{aligned}$$

Set up Area Integral:

$$\begin{aligned} A &= \int_a^b L(x)\,dx && \triangleleft \text{ from (6)} \\ &= \int_0^2 x^2 - \frac{1}{2}x + 2\,dx \\ &= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^2 \Big|_0^2 && \triangleleft \text{ Power Rule} \end{aligned}$$

$$= \frac{8}{3} - 1 = \frac{5}{3}$$ ◁ Evaluate!

Presentation of Answer: $\boxed{A = \frac{5}{3}}$.

■ Question. In the solution above, we observed *graphically* that $f(x) = \frac{1}{2}x - 1 < x^2 + 1 = g(x)$, for $0 \leq x \leq 2$. This fact was used to correctly remove the absolute values in equation (A-3). Use *sightless* or *nongraphical methods* to prove this inequality. (Use differential calculus!)

Exercise 2.3. ■

2.4. From the specification that we are working in the *st*-axis system, the axis of abscissas is therefore the *s*-axis and the axis of ordinates is the *t*-axis. Label the endpoints of the rule as

$$P(\, s_1, t_1 \,), \quad Q(\, s_2, t_2 \,).$$

The length of the horizontal rule is the "abscissa of the right-most point minus the abscissa of the left-most point." Thus,

$$\text{length} = |s_1 - s_2|. \tag{A-4}$$

Since I did not designate which of the two points lies to the left of the other, the absolute value cannot be removed.

In the quiz that follows, *think* before responding. Draw a picture to guide your thinking.

Quiz. If I had told you that "P lies to the right of Q." Then equation (A-4) can be simplified down to . . .

(a) $s_1 - s_2$ (b) $s_2 - s_1$ (c) Not enough information

A passing grade is 100%. Exercise 2.4. ■

2.5. Let's begin by graphing the three curves:

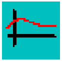


Figure A-1

$$y_1 = 3x \quad y_2 = 2x \quad x = 3.$$

I have labeled the dependent variables for easy reference.

You'll note that $\mathcal{R}$ is a triangular region; consequently, you can calculate the area of $\mathcal{R}$ using the area formula for a triangle. (Do so!)

Take the line ℓ to be the x-axis, and so the corresponding rules will be vertical. This implies that x will be the variable of integration *and* the rule function $L(x)$ will be a function of x. (Recall, in the generic set up, $L(s)$ was a function of s, where s is a real variable on the ℓ-axis.)

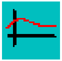


Figure A-2

Calculation of the Limits of Integration: As we move back and forth on the x-axis, drawing the corresponding rules as we go, we must ask ourselves: "What are the two extremities of this region?" Because we are using the x-axis to generate our rules, the term *extremity* will refer to the *left-most* and *right-most* extremities. You can see from Figure A-2, the left-most extremity of the region is $x = 0$ and the right-most extremity is $x = 3$.

The Lower Limit of Integration is $x = 0$.
The Upper Limit of Integration is $x = 3$.

Calculation of the Rule Function: Here is the "standard reasoning." Choose an x, $0 \le x \le 3$, and draw the corresponding vertical rule (Figure A-2). The rule extends from the line $y = 2x$ to $y = 3x$. The length of that line segment is

$$\begin{aligned} L(x) &= |y_1 - y_2| \\ &= y_1 - y_2 \qquad \triangleleft \text{ since } y_1 > y_2 \\ &= 3x - 2x = x. \end{aligned}$$

Thus,

$$L(x) = x, \quad 0 \le x \le 3. \tag{A-6}$$

This is the (vertical) rule function for this region.

Set up the Area Integral: Now, we apply the basic area formula (6) to set up the area integral,

$$\begin{aligned} A &= \int_a^b L(x)\,dx \\ &= \int_0^3 x\,dx. \qquad \triangleleft \text{ from (A-6)} \end{aligned}$$

Solve the Area Integral: From the previous line,

$$A = \int_0^3 x\,dx = \left.\frac{x^2}{2}\right|_0^3 = \frac{9}{2}$$

Take the line ℓ to be the y-axis. This implies that x will be the variable of integration *and* the rule function $L(y)$ will be a function of y. (Recall, in the generic set up, $L(s)$ was a function of s, where s is a real variable on the ℓ-axis.)

Presentation of Answer: $\boxed{A = \frac{9}{2}.}$

Exercise 2.5. ■

2.6. Graph the functions and identify the target region.

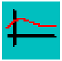


Figure A-3

$$y_1 = x^2 \quad y_2 = \sqrt{x}.$$

I have labeled the y-variable for easy reference.

Take the line ℓ to be the y-axis, and so the corresponding rules will be horizontal. This implies that y will be the variable of integration *and* the rule function $L(y)$ will be a function of y. (Recall, in the generic set up, $L(s)$ was a function of s, where s is a real variable on the ℓ-axis.)

Points of Intersection: It is easy to see that

$$\text{Points of Intersection: } (0, 0), \ (1, 1)$$

Determining the Limits of Integration: Draw a series of horizontal rules covering the region $\mathcal{R}$. You can see that the *lowest* rule is at $y = 0$ and the *highest* rule is at $y = 1$.

The Lower Limit of Integration is $y = 0$.

The Upper Limit of Integration is $y = 1$.

Calculation of the Rule Function: Draw a series of horizontal rules. No matter where you draw your horizontal rule, the left-most point is on the curve $y_2 = \sqrt{x}$ and the right-most point is on $y_1 = x^2$. For a given y, draw the rule that corresponds to y, by the horizontal rule calculation rule, equation (9), we have

$$L(y) = |x_1 - x_2|.$$

The right-hand side must be expressed in terms of y—it is not. We must strive to do so. We must take our boundary curves and solve for x.

$$\text{Curve 1:} \quad y = x^2 \implies x = \sqrt{y}$$
$$\text{Curve 2:} \quad y = \sqrt{x} \implies x = y^2.$$

Disturbingly symmetric, isn't it? Let's label the dependent variables for convenience:

$$x_1 = \sqrt{y} \qquad x_2 = y^2.$$

Looking at the graph of the region again, we see that the graph of x_1 lies to the *right* of the graph of x_2. Thus,

$$\begin{aligned} L(y) &= |x_1 - x_2| \\ &= x_1 - x_2 \\ &= \sqrt{y} - y^2. \end{aligned}$$

Thus,

$$L(y) = \sqrt{y} - y^2, \quad 0 \le y \le 1 \tag{A-8}$$

Set up the Integral:

$$\begin{aligned} A &= \int_a^b L(y)\,dy \\ &= \int_0^1 \sqrt{y} - y^2\,dy \end{aligned}$$

Calculation of the Area: From the previous line,

$$A = \int_0^1 \sqrt{y} - y^2\,dy$$

$$= \frac{2}{3}y^{3/2} - \frac{1}{3}y^3 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Presentation of Answer: $\boxed{A = \frac{1}{3}.}$

Exercise 2.6. ■

2.7. Let me outline the solution in a series of steps. These steps are the ones you should have gone through to solve this problem.

Step 1: **Draw a picture.** Sketch the graph using pencil and paper techniques, or by using your graphing calculator.

The function f is a parabola that opens up and g is a straight line. Together they enclose a region $\mathcal{R}$. Notice that for this region, the graph of g is always above the graph of f.

Step 2: **What type rule to use.** The boundary curves are written naturally as functions of x; therefore, the natural choice is to use *vertical rules* and x will be the limit of integration. Our guiding formula is then,

$$A = \int_a^b L(x)\,dx,$$

where L is the vertical rule function.

Step 3: **Find the Limits of Integration.** Follow the directions given in my point on how to find the limits of integration. It should have been clear to you that the extremities occur at the points of intersection of the two curves.

Points of Intersection: Put $f(x) = g(x)$, and solve for x.

$$\begin{aligned}(x-2)^2 - 4 &= x \\ x^2 - 4x &= x \\ x^2 - 5x &= 0 \\ x(x-5) &= 0\end{aligned}$$

Thus, the two curves intersect at $x = 0$ and $x = 5$. Let's now update our area formula:

$$A = \int_0^5 L(x)\,dx,$$

where L is the vertical rule function, yet to be determined.

Step 4: **Construct the Vertical Rule Function.** You should have reasoned as follows: The variable x is the variable of integration and the limits of integration are $0 \le x \le 5$; for each x, $0 \le x \le 5$, draw the corresponding vertical rule. The rule extends from the parabola,

$f(x) = (x-2)^2 - 4$, up to the straight line, $g(x) = x$. The length, $L(x)$, of this vertical rule is the "y-coordinate of the upper point *minus* the y-coordinate of the lower point." (See the discussion on how to calculate the length of rules.) Thus,

$$\begin{aligned} L(x) &= x - [(x-2)^2 - 4] && \triangleleft \text{ from (8)} \\ &= 5x - x^2 \end{aligned}$$

Step 5: **Set up the Area Integral.**

$$A = \int_0^5 L(x)\,dx = \int_0^5 5x - x^2\,dx$$

Step 6: **Evaluate the Area Integral.**

$$A = \int_0^5 5x - x^2\,dx$$
$$= \frac{5}{2}x^2 - \frac{1}{3}x^3\Big|_0^5$$
$$= \frac{5}{2}(5^2) - \frac{1}{3}(5^3) = \frac{125}{6}$$

Step 7: **Presentation of Solution.**

$$\boxed{A = \frac{125}{6}}$$

That's all.

Exercise 2.7. ■

2.8. *Step* 1: **Draw a picture.** Do this yourself, and identify the target region. Notice that over some portions of the region $f(x) = \sin(x)$ is below $g(x) = \cos(x)$, and over other portions, g is below f.

Step 2: **Horizontal or Vertical Rules?** The boundary curves

$$f(x) = \sin(x) \qquad g(x) = \cos(x) \qquad 0 \leq x \leq \frac{\pi}{2},$$

are naturally expressible as function of x; based on the above remarks, we shall be using *vertical rules* and x will be the variable of integration.

Step 3: **Limits of Integration.** It was given in the problem, $0 \leq x \leq \pi/2$. These two endpoints will be the *limits of integration*:

$$A = \int_0^{\pi/2} L(x)\, dx.$$

This represents the set up of the area integral based on the information so far. The only thing we need to do is to determine the *vertical rule function.*

Step 4: **Determine the (Vertical) Rule Function.** The simple answer to this is

$$L(x) = |\sin(x) - \cos(x)|,$$

however, in anticipation of the need to solve the area integral, a greater understanding of $L(x)$ is necessary. These two curves interchange positions. We need to find at what value of x do they do this.

Point of Intersection: Put $f(x) = g(x)$, and solve for x, for x only between 0 and $\pi/2$. Set

$$\sin(x) = \cos(x) \qquad 0 \le x \le \frac{\pi}{2}$$

and obtain,

$$x = \frac{\pi}{4}.$$

Thus,

$$\text{Point of Intersection: } (\,\pi/4, \sqrt{2}/2\,)$$

Because the relative positions of f and g change, ($f < g$ on $(0, \pi/4)$ and $g < f$ on $(\pi/4, \pi/2)$) we are forced to calculate $L(x)$ separately over each of these intervals.

Case 1: $0 \leq x \leq \pi/4$. Choose any x in that interval and draw the corresponding vertical rule. The rule extends from the graph of f to the graph of g. (f is below g in this case.) Therefore,

$$L(x) = |f(x) - g(x)| = |\sin(x) - \cos(x)| = \cos(x) - \sin(x).$$

(The length of a vertical rule is the ordinate of the upper curve minus the ordinate of the lower curve.)

Case 2: $\pi/4 \leq x \leq \pi/2$. Choose any x in that interval and draw the corresponding vertical rule. The rule extends from the graph of g up to the graph of f. (g is below f in this case.) Therefore,

$$L(x) = |f(x) - g(x)| = |\sin(x) - \cos(x)| = \sin(x) - \cos(x).$$

(The length of a vertical rule is the ordinate of the upper curve minus the ordinate of the lower curve.)

Summary:

$$L(x) = \begin{cases} \cos(x) - \sin(x) & 0 \leq x \leq \pi/4 \\ \sin(x) - \cos(x) & \pi/4 \leq x \leq \pi/2 \end{cases}$$

You can see that because our two boundary curves changed positions, our rule function is a *piecewise-defined function*.

Step 5: **Set up the Area Integral.** Take our rule function and substitute it back into our area integral.

$$\begin{aligned} A &= \int_0^{\pi/2} L(x)\,dx \\ &= \int_0^{\pi/4} L(x)\,dx + \int_{\pi/4}^{\pi/2} L(x)\,dx \\ &= \int_0^{\pi/4} \cos(x) - \sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\,dx \end{aligned}$$

Here, it is standard technique, that when the integrand, $L(x)$, is a piecewise defined function, the interval of integration, $[\,0, \pi/2\,]$, is broken up into subintervals corresponding to the piecewise definition of the integrand. In our case, $L(x)$ had different defining formulas over the intervals $[\,0, \pi/4\,]$ and $[\,0, \pi/2\,]$. Thus,

$$\boxed{A = \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\, dx.}$$

Step 6: **Solve the Area Integral.** I leave it to you to complete the task of solving the area problem; solve it *exactly*: a calculator is not needed here.

Step 7: **Presentation of Answer:** $\boxed{A = 2(\sqrt{2} - 1).}$

Exercise 2.8. ■

2.9. The first step is to make a rough sketch of the two functions. The left and right most extremity of the region is determined by the points of intersection of the two curves.

Points of Intersection: Set $K(x) = H(x)$, and solve for x.

$$x^2 = \frac{2}{1+x^2} \qquad \triangleleft \text{ equate them}$$

$$x^2 + x^4 = 2 \qquad \triangleleft \text{ cross-multiply}$$

$$x^4 + x^2 - 2 = 0 \qquad \triangleleft \text{ transpose}$$

$$(x^2 - 1)(x^2 + 2) = 0 \qquad \triangleleft \text{ factor it!}$$

and so,

$$x = \pm 1. \qquad \triangleleft \text{ done}$$

Therefore, the limits of integration will be -1 to 1.

Set up Area Integral: As x varies from -1 to 1, the graph of H is above the graph of K; therefore,

$$\begin{aligned} A &= \int_{-1}^{1} H(x) - K(x)\,dx \\ &= \int_{-1}^{1} \frac{2}{1+x^2} - x^2\,dx \end{aligned}$$

The Minimal Answer: $\boxed{A = \int_{-1}^{1} \frac{2}{1+x^2} - x^2\,dx.}$

The reason I only asked for the *Set up* of the integral that we do not know how to integrate (i.e., to find an antiderivative) the function H. However, the answer can be improved a little. First, we can observe that the two functions H and K are *symmetric* with respect to the y-axis; therefore,

$$A = 2\int_{0}^{1} \frac{2}{1+x^2} - x^2\,dx.$$

Even though we can't calculate the indefinite integral of the first function (at least until *Calculus II*), we can compute the integral of the second function.

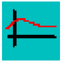
$$\begin{aligned} A &= 2\int_0^1 \frac{2}{1+x^2} - x^2\,dx \\ &= 2\int_0^1 \frac{2}{1+x^2}\,dx - 2\int_0^1 x^2\,dx \\ &= 4\int_0^1 \frac{1}{1+x^2}\,dx - \frac{2}{3} \end{aligned}$$

The best we can do at this time is to say,

$$\boxed{A = 4\int_0^1 \frac{1}{1+x^2}\,dx - \frac{1}{3}.} \tag{A-9}$$

Exercise 2.9. ■

2.10. *Step* 1: **Sketch the graph.**



$$x_1 = f(y) = y^4 \qquad x_2 = g(y) = (y+2)^2$$

Figure A-4

I have labeled the dependent variables for convenience—perhaps you did to?

Step 2: **Determine the type of rule.** The boundary curves are written as functions of y; therefore, we shall use *horizontal rules* and use y as the *variable of integration*. Therefore,

$$A = \int_a^b L(y)\,dy,$$

where L is the horizontal rule function, yet to be determined.

Step 3: **Determine the limits of integration.** It is clear from the picture of the region that the upper and lower extremities of the region occur at the points of intersection of the two boundary curves.

Points of Intersection: Put $f(y) = g(y)$ and solve for y.

$$\begin{aligned} y^4 &= (y+2)^2 \\ y^4 - (y+2)^2 &= 0 \\ [y^2 - (y+2)][y^2 + (y+2)] &= 0 \\ (y^2 - y - 2)(y^2 + y + 2) &= 0 \\ (y+1)(y-2)(y^2 + y + 2) &= 0 \end{aligned}$$

Therefore, these two curves intersect at $y = -1$ and $y = 2$. The third factor, $y^2 + y + 2$ *does not factor* because its *discriminant* is negative. These are the limits of integration; update the area integral:

$$A = \int_{-1}^{2} L(y)\, dy.$$

Step 4: **Determine the Rule Function.** For each y, $-1 \le y \le 2$, you should have drawn a (typical) horizontal rule. This rule would extend from the curve $x_1 = f(y) = y^3$ on the left over to the curve $x_2 = g(y) = (y+2)^2$ on the right. Therefore,

$$L(y) = |x_1 - x_2| = x_2 - x_1 = (y+2)^2 - y^4 \qquad \triangleleft \text{ from (9)}$$

That is to say, the length of the horizontal rule is the "right-most point minus the left-most point."

Step 5: **Set up the Area Integral.** $A = \displaystyle\int_{-1}^{2} (y+2)^2 - y^4\, dy$

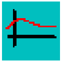
Step 6: **Solve the Area Integral.**

$$\begin{aligned} A &= \int_{-1}^{2} (y+2)^2 - y^4\, dy \\ &= \frac{1}{3}(y+2)^3\Big|_{-1}^{2} - \frac{1}{5}y^5\Big|_{-1}^{2} \\ &= \frac{63}{3} - \frac{33}{5} = \frac{122}{5} = 24\tfrac{2}{5} \end{aligned}$$

Step 7: **Presentation of Answer:** $\boxed{A = 24\tfrac{2}{5}.}$

Exercise 2.10. ■

2.11. *Step* 1: **Sketch the graph.**



$$x_1 = y^2 - 6y \qquad x_2 = |y - 1|.$$

Figure A-5

Step 2: **Determine the type of rule to use.** The boundary equations are written naturally as functions of y; therefore, horizontal rules are the natural choice. This implies that y will be the variable of integration. The basic area formula tells us that

$$A = \int_a^b L(y)\, dy$$

Step 3: **Determine the limits of integration.** This is a bit trickier, but nothing that cannot be overcome with increased concentration. A peak at the region shows that the upper and lower extremities occurs at points of intersection between the two boundary curves.

Points of Intersection: Put $x_1 = x_2$ and solve for y.

$$y^2 - 6y = |y - 1|. \tag{A-10}$$

We can't solve this equation until we "strip away" the absolute value.

Case 1: $y \geq 1$. Look for a solution, y, to (A-10) greater than 1. In this case, $|y-1| = y-1$. Thus,

$$y^2 - 6y = y - 1 \qquad \triangleleft \text{ for } y \geq 1$$

$$y^2 - 7y + 1 = 0$$

$$y = \frac{7 \pm \sqrt{49 - 4(1)(1)}}{2} \qquad \triangleleft \text{ quadratic formula}$$

Thus,

$$y = \frac{7 + \sqrt{45}}{2} \qquad \triangleleft \text{ since we want } y \geq 1$$

$$y = \frac{1}{2}(7 + 3\sqrt{5}) \qquad \triangleleft \text{ to be true to alg. roots!}$$

Thus, one point of intersection is at

$$y = \frac{1}{2}(7 + 3\sqrt{5}) \tag{A-11}$$

Case 2: $y \leq 1$. Look for a solution, y, to (A-10) less than 1. In this case, $|y - 1| = 1 - y$. Thus,

$$\begin{aligned} y^2 - 6y &= 1 - y && \triangleleft \text{ for } y \leq 1 \\ y^2 - 5y - 1 &= 0 \\ y &= \frac{5 \pm \sqrt{25 - 4(1)(-1)}}{2} && \triangleleft \text{ quadratic formula} \end{aligned}$$

Thus,

$$y = \frac{5 - \sqrt{29}}{2} \qquad \triangleleft \text{ since we want } y \leq 1$$

Our second point of intersection is then

$$y = \frac{1}{2}(5 - \sqrt{29}) \tag{A-12}$$

The two y-values in (A-11) and (A-12) are upper and lower limits of integration, respectively. These are such awkward quantities, I'll just label them symbolically, let

$$a = \frac{1}{2}(5 - \sqrt{29}) \qquad b = \frac{1}{2}(7 + 3\sqrt{5}) \tag{A-13}$$

No need to write these hideous expressions over and over again. I hope you thought of doing the same thing.

Step 4: **Find the Rule function.** Look at Figure A-5. For any y, $a \le y \le b$, draw a horizontal rule. The rule extends from $x_1 = y^2 - 6y$ on the left, over to $x_2 = |y - 1|$ on the right. Therefore,

$$L(y) = x_2 - x_1 = |y - 1| - (y^2 - 6y)$$

Step 5: **Set up the Area Integral.**

$$A = \int_a^b |y - 1| - y^2 + 6y \, dy$$

We can't really evaluate this integral until we remove the absolute value. Thus,

$$\begin{aligned}
A &= \int_a^b |y-1| - y^2 + 6y\,dy \\
&= \int_a^1 |y-1| - y^2 + 6y\,dy + \int_1^b |y-1| - y^2 + 6y\,dy \\
&= \int_a^1 1 - y - y^2 + 6y\,dy + \int_1^b y - 1 - y^2 + 6y\,dy \\
&= \int_a^1 1 + 5y - y^2\,dy + \int_1^b -1 + 7y - y^2\,dy
\end{aligned}$$

The last integral is a proper presentation of the area integral—all set up and ready to be evaluated.

$$\boxed{A = \int_a^1 1 + 5y - y^2\,dy + \int_1^b -1 + 7y - y^2\,dy}$$

where, from (A-13)

$$a = \frac{1}{2}(5 - \sqrt{29}) \qquad b = \frac{1}{2}(7 + 3\sqrt{5})$$

Did you get it?

This is simple, but tedious, to evaluate . . . I leave it to you `:-{)`.

Exercise 2.11. ■

2.12. *Vertical Rules*: Using vertical rules, the set up is simple:

$$A = \int_0^{49} \sqrt[3]{1 + \sqrt{x}}\, dx.$$

This is a nasty looking one.

Horizontal Rules: This is a more "unnatural choice" but leads to a more mathematically tractible integral. The basic set up is

$$A = \int_0^{2} L_h(y)\, dy,$$

where L_h is the horizontal rule function. Do you see where I got the upper limit of 2?

The Rule Function: Since the rule function is a function of y, our boundary curves must be expressed as functions of y.

$$y = \sqrt[3]{1 + \sqrt{x}} \implies y^3 = 1 + \sqrt{x} \implies \sqrt{x} = y^3 - 1$$
$$\implies x = (y^3 - 1)^2$$

Now, the function function can be easily found.

$$L_h(y) = \begin{cases} 49 & \text{for } 0 \le y \le 1 \\ 49 - (y^3 - 1)^2 & \text{for } 1 \le y \le 2 \end{cases}$$

Set up of Area Integral:

$$\begin{aligned} A &= \int_0^2 L(y)\,dy \\ &= \int_0^1 49\,dy + \int_1^2 49 - (y^3 - 1)^2\,dy \\ &= \int_0^2 49\,dy - \int_1^2 (y^3 - 1)^2\,dy \qquad \triangleleft \text{ What did I do here?} \\ &= 2(49) - \int_1^2 (y^3 - 1)^2\,dy \end{aligned}$$

I leave the evaluation of the integral to you. I computed it to be . . .

Presentation of Answer: $\boxed{A = \dfrac{1209}{14}.}$

Did you get it?

Exercise 2.12. ■

2.13. Make a rough sketch of the graph.

Vertical Rules: The boundary curves must be written as functions of x:

$$y_1 = \frac{1}{49}x + 1 \qquad y_2 = (1 + \sqrt{x})^{1/3}$$

For $0 \le x \le 49$, $y_1 \le y_2$, i.e., the line is *below* the other curve.

For any x, $0 \le x \le 49$,

$$\begin{aligned} L(x) &= |y_1 - y_2| = y_2 - y_1 \qquad \triangleleft \text{ from (8)} \\ &= \frac{1}{49}x + 1 - (1 + \sqrt{x})^{1/3}. \end{aligned}$$

Thus,

$$\boxed{A = \int_0^{49} \frac{1}{49}x + 1 - (1 + \sqrt{x})^{1/3}\, dx.}$$

This ugly looking integral would be very difficult to solve indeed.

Let's set up the area integral using y as the variable of integration.

Horizontal Rules: As y varies, draw the corresponding horizontal rules. The lower-most extremity of the region occurs at $y = 1$ and the upper most extremity occurs at $y = 2$. (Why?) Thus,

$$A = \int_1^2 L(y)\,dy, \tag{A-14}$$

where, $L(y)$ is the horizontal rule function, yet to be determined.

Express the boundary curves as functions of y:

$$\begin{aligned} x - 49y + 49 &= 0 \\ x &= 49y - 49 \\ x &= 49(y-1) \end{aligned} \qquad \begin{aligned} y &= (1+\sqrt{x})^{1/3} \\ y^3 &= 1+\sqrt{x} \\ \sqrt{x} &= y^3 - 1 \\ x &= (y^3-1)^2 \end{aligned}$$

Let's label the dependent variable for convenience. The boundary curves are given by

$$x_1 = 49y - 49 \qquad x_2 = (y^3-1)^2$$

Find the horizontal rule function: For any y, $1 \le y \le 2$, draw the corresponding rule. It extends from the curve $x_2 = (y^3 - 1)^2$ over to the line $x_1 = 49(y - 1)$.

$$\begin{aligned} L(y) &= \text{right-most absicssa minus left-most abscissa} \\ &= 49(y - 1) - (y^3 - 1)^2 \qquad \text{(A-15)} \end{aligned}$$

Set up the area integral: From (A-14) and (A-15)

$$A = \int_1^2 L(y)\,dy = \int_1^2 49(y - 1) - (y^3 - 1)^2\,dy$$

Solve the Area Integral:

$$
\begin{aligned}
A &= \int_1^2 49(y-1) - (y^3-1)^2\,dy \\
&= \frac{49}{2}\,(y-1)^2\Big|_1^2 - \int_1^2 y^6 - 2y^3 + 1\,dy \\
&= \frac{49}{2} - \left.\frac{1}{7}y^7 - \frac{1}{2}y^4 + y\right|_1^2 \\
&= \frac{49}{2} - \frac{1}{7}(2^7-1) + \frac{1}{2}(2^4-1) - (2-1) \\
&= \frac{49}{2} - \frac{127}{7} + \frac{15}{2} - 1 \\
&= 12\tfrac{6}{7}
\end{aligned}
$$

Presentation of Answer: $\boxed{A = 12\tfrac{6}{7}.}$

Exercise 2.13. ■

2.14. *Step 1*: **Draw a picture.** Done!

Step 2: **Vertical or Horizontal Rules.** Horizontal—forced on us!

Step 3: **Limits of Integration.** Draw a series of horizontal rules over the target region given. Notice the lowest extremity of the region occurs at the rule generated by $y = 0$. The upper extremity of the region occurs at the horizonal rule that passes through the intersection of $f(x) = 1/x^2$ and $g(x) = 8x$.

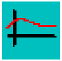


Figure A-6

Points of Intersection: Put $f(x) = g(x)$ and solve for x.

$$\frac{1}{x^2} = 8x \implies \frac{1}{8} = x^3 \implies x^3 = \frac{1}{8} \implies x = \frac{1}{2}$$

Note that $f(\frac{1}{2}) = g(\frac{1}{2}) = 4$; therefore, the two curves intersect at $(\frac{1}{2}, 4)$.

What are the limits of integration? They are $y = 0$ to $y = 4$, the latter being the horizontal position of the upper extremity of the target region. Thus,

$$A = \int_0^4 L(y)\,dy$$

where $L(y)$ is the horizontal rule function yet to be determined. Note that the variable of integration is the y-variable.

Step 4: **Calculate the Rule Function.** We are using horizontal rules. Draw a series of rules over the target region. You'll notice that some of them extend from a point on the graph of $g(x) = 8x$ to a point on the graph of $f(x) = 1/x^2$, while others extend from a point on $g(x) = 8x$ over to the vertical line at $x = 2$. We see then that the rule function changes definitions; it will be a piecewise defined function.

For horizontal rules, the boundary curves need to be expressed in terms of y. So we solve for x in terms of y:

$$y = f(x) = \frac{1}{x^2} \implies x = \frac{1}{\sqrt{y}}$$
$$y = g(x) = 8x \implies x = \frac{y}{8}$$

Let's label the dependent variables for easy reference:

$$x_1 = \frac{1}{\sqrt{y}} \qquad x_2 = \frac{y}{8}$$

Case 1: For any given y, $0 \le y \le \frac{1}{4}$, the horizontal rule extends from $x_2 = y/8$ over to the vertical line at $x = 2$; therefore,

$$\begin{aligned} L(y) &= \left| \frac{y}{8} - 2 \right| \\ &= 2 - \frac{y}{8} \qquad 0 \le y \le \frac{1}{4}, \end{aligned}$$

since $2 > y/8$ over the interval $0 \le y \le \frac{1}{4}$.

Case 2: For any given y, $\frac{1}{4} \le y \le 4$, the horizontal rule extends from $x_2 = y/8$ over to $x_1 = 1/\sqrt{y}$; therefore,

$$\begin{aligned} L(y) &= \left| \frac{y}{8} - \frac{1}{\sqrt{y}} \right| \\ &= \frac{1}{\sqrt{y}} - \frac{y}{8} \qquad \frac{1}{4} \le y \le 4. \end{aligned}$$

Finally, the horizontal rule function is

$$L(y) = \begin{cases} 2 - \dfrac{y}{8} & 0 \le y \le \frac{1}{4} \\ \dfrac{1}{\sqrt{y}} - \dfrac{y}{8} & \frac{1}{4} \le y \le 4 \end{cases}$$

Step 5: **Set up the Area Integral.**

$$A = \int_0^4 L(y)\,dy$$

$$= \int_0^{1/4} L(y)\,dy + \int_{1/4}^{4} L(y)\,dy$$

$$= \int_0^{1/4} 2 - \frac{y}{8}\,dy + \int_{1/4}^{4} \frac{1}{\sqrt{y}} - \frac{y}{8}\,dy$$

Step 6: **Evaluation of Same.**

$$A = \int_0^{1/4} 2 - \frac{y}{8}\,dy + \int_{1/4}^{4} \frac{1}{\sqrt{y}} - \frac{y}{8}\,dy$$

$$= \int_0^{1/4} 2 - \frac{y}{8}\,dy + \int_{1/4}^{4} y^{-1/2} - \frac{y}{8}\,dy$$

$$= \left(2y - \frac{y^2}{16}\right)\Bigg|_0^{1/4} + \left(2y^{1/2} - \frac{y^2}{16}\right)\Bigg|_{1/4}^{4} = \frac{5}{2}$$

Presentation of Answer: $\boxed{A = \frac{5}{2}}$

Exercise 2.14. ■

2.15. In this case, *vertical rules* would be lead to the simplest set up. Compare vertical rules to horizontal rules for this region. The vertical rules always span the region from the parabola up to the circle. Horizontal rules in places span the region from the left-hand side of the parabola to the right-hand side; elsewhere, if spans from the left-hand side of the circle to the right-hand side. Obviously, the horizontal rule function will have piecewise definition. Given these observation and the fact that in the equation $x^2 + y^2 = 1$ either variable can be solved for with equal ease, the obvious choice is to use vertical rules. So,

$$A = \int_a^b L_v(x)\,dx,$$

where a and b are the limits of integration and L_v is the vertical rule function. Note that x will be the variable of integration.

Points of Intersection. We find the points of intersection for the x-coordinates will be our limits of integration. At the points of intersection we have

$$y = x^2 \text{ and } x^2 + y^2 = 2 \implies y + y^2 = 2.$$

As the fates would have it, this last equation factors:

$$y^2 + y - 2 = 0 \implies (y-1)(y+2) = 0$$

The solutions are $y = 1$ and $y = -2$. Note that for our setting, $y \geq 0$ (since, for example, $y = x^2 \geq 0$). Throw out the solution $y = -2$ and we get a unique solution, $y = 1$. To find the x-coordinates of the points of intersection, we "plug" the value of $y = 1$ back into any of the two equations—I'll choose the simplier one. Substitute $y = 1$ into $y = x^2$ to get $x^2 = 1$, or $x = \pm 1$.

The points of intersection are then $(-1, 1)$ and $(1, 1)$.

Our basic area integral has the form: $\displaystyle\int_{-1}^{1} L_v(x)\, dx$.

The vertical rule function is easy to calculate. We must write the boundary curves as functions of x. The circle is always above the parabola so

$$L_v(x) = \sqrt{2 - x^2} - x^2$$

Final Set up: $\boxed{A = \int_{-1}^{1} \sqrt{2 - x^2} - x^2 \, dx}$

Exercise Notes: The above form can be simplified a little using the symmetry of the integrand:

$$A = 2 \int_{0}^{1} \sqrt{2 - x^2} - x^2 \, dx$$

- The above integral can be partially solved:

$$\begin{aligned} A &= 2 \int_{0}^{1} \sqrt{2 - x^2} - 2 \int_{0}^{1} x^2 \, dx \\ &= 2 \int_{0}^{1} \sqrt{2 - x^2} \, dx - \frac{2}{3} \end{aligned} \tag{A-17}$$

■ The remaining integral can be solved using the techniques of *Calculus II*; alternately, the student can go to a *table of integrals* and look up the integral of $\sqrt{2 - x^2}$, then evaluate it between the limits of 0 to 1, or, the student can use a **CAS** to compute the integral. ■

Exercise 2.15. ■

2.16. Draw a sketch of the region. Compare vertical rules to horizontal rules. In this case, *horizontal rules* are the preferred. The horizontal rule function will be determined by a single formula; the vertical rule function will have a piecewise definition since in some places the vertical rules extend from the x-axis up to the parabola, and in other places it extends from the x-axis to the circle.

In Exercise 2.15 we determined that the points of intersection between the two curves is

$$\text{Points of Intersection: } (-1, 1), (1, 1)$$

Limits of Integration: Using horizontal rules, it is clear from the picture of the region that

$$A = \int_0^1 L_h(y)\, dy.$$

The lower extremity of the region is $y = 0$ and the upper extremity is $y = 1$.

Horizontal Rule Function: For any given y, $0 \le y \le 1$, draw the corresponding rule. Observe that the right-most point is on the parabola and the left-most lies on the circle. Therefore,

$$L(y) = x_2 - x_1$$

where x_2 is the x-coordinate on the circle and x_1 is the x-coordinate on the parabola:

$$\text{Parabola: } y = x^2 \qquad \text{Circle: } x^2 + y^2 = 2$$

$$x_1 = \sqrt{y} \qquad\qquad x_2 = \sqrt{2 - y^2}$$

Keep in mind that $y \ge 0$. This is the reason why the above square roots are not prefixed by '$\pm$.' Thus,

$$L(y) = x_2 - x_1 = \sqrt{2 - y^2} - \sqrt{y}.$$

Set up Area Integral:

$$\boxed{A = \int_0^1 \sqrt{2 - y^2} - \sqrt{y}\, dy}$$

Exercise Notes: The above integral can be partially solved:

$$A = \int_0^1 \sqrt{2 - y^2}\, dy - \frac{2}{3}.$$

The remaining integral can be solved using techniques from *Calculus II*, or by looking up this integral in a *table of integration*, or by using a **CAS**, a computer algebraic system such as **Maple**. ∎

• Continuation. According to my CRC STANDARD MATHEMATICAL TABLES:

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right],$$

where $\sin^{-1}$ is the *inverse sine function*. This function is programmed on many calculators. Your assignment, should you accept it, is to continue the calculation using this fact.

Exercise 2.16. ∎

Solutions to Examples

2.1. A quick sketch of the graphs of these two functions shows that

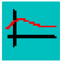


Figure S-1

$$g(x) = x^2 \leq x = f(x) \quad 0 \leq x \leq 1$$

Notice that the region bounded by these two functions *exactly* conforms to the region described in Theorem 2.1. In our example, the graph of f is always above the graph of g.

From the area formula (5) we have,

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

In our problem, $f(x) = x$, $g(x) = x$, $a = 0$, and $b = 1$. Thus,

$$A = \int_0^1 |x - x^2|\, dx. \tag{S-1}$$

To evaluate this definite integral, we must remove the absolute values (legally). As was noted above already, $g(x) \leq f(x)$ over the interval $[\,0,1\,]$; i.e., $x^2 \leq x$ over $[\,0,1\,]$. This means

$$|x - x^2| = x - x^2 \quad 0 \leq x \leq 1$$

Now, plugging directly into our area formula, (S-1),

$$\begin{aligned} A &= \int_0^1 |x - x^2|\,dx \\ &= \int_0^1 x - x^2\,dx \\ &= \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned}$$

Answer: $\boxed{A = \dfrac{1}{6}.}$ Example 2.1. ■

2.2. Recall Figure S-1 from EXAMPLE 2.1. Again, we observe that

$$g(x) = x^2 \leq x = f(x) \quad 0 \leq x \leq 1,$$

The Rule Function: We calculate the rule function using vertical rules. For any x, $0 \leq x \leq 1$, draw the *vertical rule generated by* x. Observe that it extends from a lower altitude of $y_1 = g(x) = x^2$ to an upper altitude of $y_2 = f(x) = x$. The *length* of this vertical rule is

$$\begin{aligned} L(x) &= |\, y_1 - y_2 \,| \\ &= y_2 - y_1 && \triangleleft \text{ since } y_1 < y_2 \\ &= x - x^2 && \triangleleft \text{ write it in terms of } x \end{aligned} \tag{S-2}$$

Thus,

$$L(x) = \begin{cases} x - x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $L(x) = 0$ outside the interval $[\,0, 1\,]$, since any vertical rule outside that interval does not intersect the region of interest, so the length of the rule is zero.

Set up the Area Integral:

$$\begin{aligned} L(x) &= \int_a^b L(x)\,dx && \triangleleft \text{ from (6)} \\ &= \int_0^1 x - x^2\,dx && \triangleleft \text{ from (S-2)} \end{aligned} \tag{S-3}$$

The limits of integration are 0 to 1 since the vertical rule is zero everywhere else.

Calculation of Area: From (S-3), we have

$$\begin{aligned} A &= \int_0^1 x - x^2\,dx \\ &= \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 \\ &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned}$$

Presentation of Answer: $\boxed{A = \frac{1}{6}.}$

Example Notes: The idea of the solution is conceptually quite simple. We compute the vertical rule function, $L(x) = x - x^2$. For any x, $0 \le x \le 1$, $L(x)$ is the length of the cross-section at x.

▪ Having obtained the function that calculates the distance across the region at any given point, we then integrate it

$$A = \int_0^1 L(x)\,dx = \int_0^1 x - x^2\,dx$$

from the left-most extremity of the region, that's $x = 0$, over to the right-most extremity of the region, that's $x = 1$. ■

Example 2.2. ■

2.3. Let's begin by graphing the two curves:

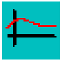


Figure S-2

$$y_1 = x^2 - 2x - 3 \quad y_2 = 3x - 7.$$

I have renamed the dependent variable for easy reference. Complete the square of y_1: $y_1 = (x-1)^2 - 4$. This means that the graph of y_1 is a parabola that opens up and has vertex at $V(1, -4)$. y_2 is obviously a line. Together they enclose a region. (See Figure S-2.)

Take the line ℓ to be the x-axis, and so the corresponding rules will be vertical. This implies that x will be the variable of integration *and* the rule function $L(x)$ will be a function of x. (Recall, in the generic set up, $L(s)$ was a function of s, where s is a real variable on the ℓ-axis.)

Points of Intersection: Let's find the points of intersection of the two curves—these will play an important role in setting up the integral.

Set $y_1 = y_2$, and solve for x

$$x^2 - 2x - 3 = 3x - 7$$
$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$

The two curves intersect at $x = 1$ and $x = 4$. Substituting these values of x back into either y_1 or y_2, we see that the points of intersection are

$$\text{Points of Intersection: } (1, -4),\ (4, 5) \qquad \text{(S-4)}$$

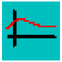


Figure S-3

Calculation of the Limits of Integration: As we move back and forth on the x-axis, drawing the corresponding vertical rules as we go, we must ask ourselves the question: "What are the two extremities of this region?" Because we are using the x-axis to generate our rules, the term *extremity* will refer to the *left-most* and *right-most* extremities. You can see from Figure S-3 that the left-most extremity of the region is $x = 1$ and the right-most extremity is $x = 4$.

The Lower Limit of Integration is $x = 1$.
The Upper Limit of Integration is $x = 4$.

Calculation of the Rule Function: Here is the "standard reasoning." Choose a x, $1 \le x \le 4$, and draw the corresponding vertical rule (Figure S-3). The rule extends from the parabola $y_1 = x^2 - 2x - 3$ to the line $y_2 = 3x - 7$. For any x, $1 \le x \le 4$, the length of that line segment is

$$\begin{aligned} L(x) &= |y_1 - y_2| = y_2 - y_1 && \triangleleft \text{ since } y_2 > y_1 \\ &= (3x - 7) - (x^2 - 2x - 3) \\ &= -x^2 + 5x - 4. \end{aligned}$$

Thus,

$$L(x) = -x^2 + 5x - 4. \tag{S-5}$$

This is the (vertical) rule function for this region.

Set up the Area Integral: Now, we apply the basic area formula (6) to set up the area integral; from (S-5) we have,

$$\begin{aligned} A &= \int_a^b L(x)\,dx \\ &= \int_1^4 -x^2 + 5x - 4\,dx. \end{aligned}$$

Solve the Area Integral: From the previous line,

$$A = \int_1^4 -x^2 + 5x - 4\,dx = -\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\Big|_1^4 = \frac{9}{2} \quad \triangleleft \text{ Verify!}$$

Presentation of Answer: $\boxed{A = \dfrac{9}{2}.}$ (S-6)

Example 2.3. ■

2.4. Rename the dependent variables for easy reference,

$$y_1 = x^2 - 2x - 3 \quad y_2 = 3x - 7,$$

and refer to Figure S-2.

Points of Intersection: As was found (S-4) in the solution to EXAMPLE 2.3:

$$\text{Points of Intersection: } (\,1, -4\,),\ \ (\,4, 5,\ )$$

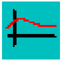


Figure S-4

Calculation of the Limits of Integration: As we move up and down on the y-axis, drawing the corresponding horizontal rules as we go, we must ask ourselves the question: "What are the two extremities of this region?" Because we are using the y-axis to generate our rules, the term *extremity* will refer to the *upper-most* and *lower-most* extremities. You can see from Figure S-4 that the lower-most extremity of the region is $y = -4$ and the upper-most extremity is $y = 5$.

The Lower Limit of Integration is $y = -4$.
The Upper Limit of Integration is $y = 5$.

Calculation of the Rule Function: Here is the "standard reasoning." Choose a y, $-4 \le y \le 5$, and draw the corresponding vertical rule (Figure S-4). The rule extends from the line $y_2 = 3x - 7$ on the left, to the parabola $y_1 = x^2 - 2x - 3$ on the right.

The rule is horizontal (parallel to the x-axis). Distances parallel to the x-axis are measured on the x-axis. To find the length of the horizontal rule we take the x-coordinate of the right-most point and subtract the x-coordinate of the left-most point. To do this we must take the boundary curves and solve for x:

$$y = x^2 - 2x - 3 \implies x = 1 \pm \sqrt{y+1} \tag{S-7}$$

$$y = 3x - 7 \implies x = \frac{1}{3}(y+7)$$

The first representation was obtained by the quadratic formula. The $\pm$ in equation (S-7) corresponds to the two halves of the parabola. We are interested in the right-hand side of the parabola so we will take the '+' sign.

The equations that enclose region $\mathcal{R}$ can be written as

$$x_1 = 1 + \sqrt{y+4} \qquad x_2 = \frac{1}{3}(y+7)$$

Now x is written as a function of y, and the functions are anonymous (they have no name) so I have labeled the dependent variables (the x's) for easy reference.

This enables us to calculate the rule function: For any y, $-4 \leq y \leq 5$, we have,

$$\begin{aligned} L(y) &= |x_1 - x_2| = x_1 - x_2 && \triangleleft \text{ since } x_1 > x_2 \\ &= (1 + \sqrt{y+4}) - (\frac{y+7}{3}), \end{aligned}$$

Thus,

$$L(y) = (1 + \sqrt{y+4}) - (\frac{y+7}{3}) \quad -4 \leq y \leq 5 \tag{S-8}$$

This is the (horizontal) rule function for this region.

Set up the Area Integral: Now, we apply the basic area formula (6) to set up the area integral,

$$\begin{aligned} A &= \int_a^b L(y)\,dy \\ &= \int_{-4}^{5} 1 + \sqrt{y+4} - \frac{y+7}{3}\,dy. \end{aligned}$$

Solve the Area Integral: From the previous line,

$$\begin{aligned} A &= \int_{-4}^{5} 1 + \sqrt{y+4} - \frac{1}{3}(y+7)\,dy \\ &= y + \frac{2}{3}(y+4)^{3/2} - \frac{1}{6}(y+7)^2 \Big|_{-4}^{5} \\ &= \frac{9}{2} \end{aligned}$$

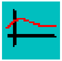
◁ Verify all details!

Presentation of Answer: $\boxed{A = \dfrac{9}{2}}$

Example Notes: The answer just given is the same one presented in equation (S-6) of EXAMPLE 2.3. At least in these two examples, the *Method of Rule* yielded the same answer. ■

Example 2.4. ■

2.5. *Step* 1: **Draw a picture.** The figure to the left represents the graph of the two functions



$$y_1 = 4 - x^2 \qquad y_2 = 1 - 2x.$$

Figure S-5

Looking at Figure S-5 should make it clear to you the region whose area we are attempting to calculate.

By the way, notice that I have used the technique of labeling the dependent variables differently in order to give these anonymous functions names. (See anonymous functions)

Step 2: **Vertical or Horizontal?** The boundary curves are written as functions of x, therefore, the use of vertical rules is the natural choice; consequently, our variable of integration is x. The *form* of the area integral is

$$A = \int_a^b L(x)\, dx, \qquad \triangleleft \text{ from (6)}$$

where L is the vertical rule function (yet to be determined), and the limits of integration, a and b, are not given—we must find them. Notice that I have use x as the variable of integration—this is called for since we are using vertical rules, and the x-axis is the horizontal axis.

Step 3: **Find the Limits of Integration.** You'll note that from Figure S-5, the left-most extremity and the right-most extremity of the region occur at the points of intersection between the two curves y_1 and y_2. The first step, then, is to find where the two curves cross each other; the limits of integration will be the *abscissas of intersection*!

Points of Intersection: To find where the two curves intersect, we equate their *ordinates*:

$$y_1 = y_2$$

$$4 - x^2 = 1 - 2x$$

$$x^2 - 2x - 3 = 0, \qquad (x + 1)(x - 3) = 0, \qquad x = -1, 3$$

Result: The two curves intersect at $x = -1, 3$. These are the left-most and right-most extremities of the region and will, therefore, be the limits of integration:

$$A = \int_{-1}^{3} L(x)\, dx,$$

where L is the vertical rule function, still to be determined.

Step 4: **Calculation of the Rule Function.** Figure S-5 also shows that y_1 is always *above* y_2. Therefore, it is easy to calculate the rule function. For any x, $-1 \le x \le 3$, draw the vertical rule generated by this x. The length of this rule is

$$\begin{aligned} L(x) &= |y_1 - y_2| && \triangleleft \text{ from (8)} \\ &= y_1 - y_2 && \triangleleft \text{ since } y_1 > y_2 \\ &= (4 - x^2) - (1 - 2x) && \triangleleft \text{ substitute} \end{aligned}$$

Thus,

$$L(x) = 3 + 2x - x^2 \qquad -1 \le x \le 3.$$

Step 5: **Set up the Area Integral.**

$$A = \int_{-1}^{3} L(x)\,dx = \int_{-1}^{3} 3 + 2x - x^2\,dx$$

Step 6: **Solve the Area Integral.**

$$\begin{aligned} A &= \int_{-1}^{3} 3 + 2x - x^2\,dx \\ &= 3x + x^2 - \frac{x^3}{2}\Bigg|_{-1}^{3} = \frac{32}{3} \end{aligned}$$

Step 7: **Presentation of Answer:** $\boxed{A = \frac{32}{3}.}$

Example 2.5. ■

2.6. *Step* 1: **Graph the two functions**,

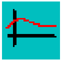


Figure S-6

$$f(x) = x \text{ and } g(x) = x^2, \quad 0 \leq x \leq 2.$$

Step 2: **What type of rule to use.** The boundary curves are written naturally as functions of x. Therefore, the most natural choice is to use *vertical rules*; consequently, the *variable of integration* will be x.

Step 3: **Determine the limits of integration.** The limits of integration will be the left-most and right-most extremities of the region. In this problem, there was a restriction on the functions to the interval $[0, 2]$. The numbers $a = 0$ and $b = 2$ will be the limits of integration since no portion of the region can exist to the left of $a = 0$ and no portion of the region can exist to the right of $b = 2$. (See remarks on the determination of the limits of integration.)

Step 4: **The Vertical Rule Function:** Based on this graph, Figure S-6, observe that

- For $0 \leq x \leq 1$, $f(x) \geq g(x)$;

- For $1 \leq x \leq 2$, $f(x) \leq g(x)$.

That is, over the interval $[\,0, 1\,]$, f is the upper-most boundary curve and g is the lower-most boundary curve. Over the interval $[\,1, 2\,]$, the positions are reversed.

These observations are necessary to compute the vertical rule functions. integrand of the formula (6).

- For $0 \leq x \leq 1$,

  $$L(x) = f(x) - g(x) = x - x^2,$$

  which represents the rule, "upper minus lower," as stated in the rule for calculating rules.

- For $1 \leq x \leq 2$,

  $$L(x) = |f(x) - g(x)| = -(f(x) - g(x)) = g(x) - f(x) = x^2 - x,$$

  again, this is "upper minus lower."

From these considerations we can formulate the vertical rule function:

$$L(x) = \begin{cases} x - x^2 & 0 \le x \le 1 \\ x^2 - x & 1 \le x \le 2 \end{cases} \tag{S-9}$$

Step 5: **Set up the Area Integral.** Now, lets set up the area integral, (6), using *good technique*:

$$\begin{aligned} A &= \int_0^2 L(x)\,dx && \triangleleft\ (6) \\ &= \int_0^1 L(x)\,dx + \int_1^2 L(x)\,dx && \triangleleft\ \text{Additive Prop.} \\ &= \int_0^1 x - x^2\,dx + \int_1^2 x^2 - x\,dx && \triangleleft\ \text{(S-9)} \end{aligned}$$

Step 6: **Evaluate the Area Integral.**

$$A = \int_0^1 x - x^2\,dx + \int_1^2 x^2 - x\,dx \tag{S-10}$$

$$= \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 + \left.\frac{x^3}{3} - \frac{x^2}{2}\right|_1^2$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

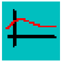
$$= \frac{1}{6} + \left[\frac{4}{6} - \frac{-1}{6}\right]$$

$$= \frac{1}{6} + \frac{5}{6} = 1$$

Step 7: **Presentation of Answer:** $\boxed{A = 1.}$

Example 2.6. ■

2.7. *Step* 1: **Sketch the graph**



$$x_1 = f(y) = y^3 \qquad x_2 = g(y) = y$$

Figure S-7

Notice that I have labeled the dependent variables for convenient reference.

Step 2: **Determine the kind of rule to use.** The boundary curves, $f(y) = y^3$ and $g(y) = y$, are written as functions of y; therefore, the *natural choice* is to use *horizontal rules* and to have y as the variable of integration. The basic formula is

$$A = \int_a^b L(y)\, dy,$$

where, $L(y)$ is the horizontal rule function, yet to be determined.

Step 3: **Determine the limits of integration.** A look at the region defined by these two curves shows that the upper- and lower-most extremities occur at the points of intersection of these two curves.

Points of Intersection: Put $f(y) = g(y)$, and solve for y.

$$\begin{aligned} y^3 &= y \\ y^3 - y &= 0 \\ y(y^2 - 2) &= 0 \\ y(y-1)(y+1) &= 0 \end{aligned}$$

Ignoring the points outside the first quadrant, we see the two curves intersect when $y = 0$ and $y = 1$. These are the limits of integration. Let's update the area integral:

$$A = \int_0^1 L(y)\, dy,$$

where, $L(y)$ is the horizontal rule function, yet to be determined.

Step 4: **Calculation of the Rule Function.** For each y, $0 \le y \le 1$, draw the corresponding horizontal rule. The left-most point of this rule is on the curve $x_1 = f(y) = y^3$ and the right-most point is on the curve $x_2 = g(y) = y$. Therefore,

$$L(y) = |x_1 - x_2| = x_2 - x_1 = y - y^3 \qquad \triangleleft \text{ from } (9)$$

That is, the "abscissa of the right-most point minus the abscissa of the left-most point."

Step 5: **Set up the Area Integral:** $A = \displaystyle\int_0^1 y - y^3\, dy$

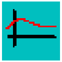
Step 6: **Solve the Area Integral:**

$$A = \int_0^1 y - y^3\, dy = \frac{1}{2}y^2 - \frac{1}{4}y^4 \bigg|_0^1 = \frac{1}{4}$$

Step 7: **Presentation of Answer:** $\boxed{A = \frac{1}{4}.}$

Example 2.7. ■

2.8. *Step* 1: **Sketch the region.**



$$x_1 = y^2 \qquad x_2 = y^3 - 4y + 4$$

Figure S-8

Where I have labeled the dependent variables for convenience.

Step 2: **Determine the type of rule used.** The first observation is that the boundary curves are written as functions of y. Therefore, the natural choice is to use y as the variable of integration and horizontal rules.

Step 3: **Find the limits of integration.** These are found at the points of intersection of the two curves.

Points of Intersection: Put $x_1 = x_2$ and solve for y.

$$\begin{aligned} y^3 - 4y + 4 &= y^2 \\ y^3 - y^2 - 4y + 4 &= 0 && \triangleleft \text{ transpose} \\ (y^3 - 4y) - (y^2 - 4) &= 0 && \triangleleft \text{ tricky rearrangement} \\ y(y^2 - 4) - (y^2 - 4) &= 0 && \triangleleft \text{ routine factorization} \\ (y - 1)(y^2 - 4) &= 0 && \triangleleft \text{ factor out } y^2 - 4 \\ (y - 1)(y - 2)(y + 2) &= 0 && \triangleleft \text{ factor again} \end{aligned}$$

The points of intersection are $y = 1, 2, -2$. (The above is one of those tricky little factorizations students hate so much.)

Limits of Integration: The lower-most extremity of the region is $y = -2$ and the upper-most extremity is $y = 2$. Therefore,

$$A = \int_{-2}^{2} L(y)\, dy$$

where, $L(y)$ is the horizontal rule function, yet to be determined. The other point of intersection, $y = 1$, is a point at which the two curves interchange relative positions.

Step 4: **The Rule Function.** The two boundary curves intersect three times. Two of the intersection points form the two extremities of the region, the one the third one, $y = 1$, is a point at which the boundary curves interchange positions.

Case 1: For $-2 \leq y \leq 1$, the function x_1 lies to the left of the function x_2; therefore,

$$\begin{aligned} L(y) &= x_2 - x_1 && \triangleleft \text{ from } (9) \\ &= (y^3 - 4y + 4) - y^2 \\ &= y^3 - y^2 - 4y + 4 \end{aligned}$$

Case 2: For $1 \le y \le 2$, the function g lies to the left of the function f; therefore,

$$
\begin{aligned}
L(y) &= x_1 - x_2 \qquad \triangleleft \text{ from } (9) \\
&= y^2 - (y^3 - 4y + 4) \\
&= -y^3 + y^2 + 4y - 4 \\
&= -(y^3 - y^2 + 4y + 4)
\end{aligned}
$$

Thus,

$$
L(y) = \begin{cases} y^3 - y^2 - 4y + 4 & -2 \le y \le 1 \\ -(y^3 - y^2 + 4y + 4) & 1 \le y \le 2 \end{cases}
$$

Step 5: **Set up the Area Integral.**

$$\begin{aligned} A &= \int_{-2}^{2} L(y)\,dy \\ &= \int_{-2}^{1} L(y)\,dy + \int_{1}^{2} L(y)\,dy \\ &= \int_{-2}^{1} y^3 - y^2 - 4y + 4\,dy + \int_{1}^{2} -(y^3 - y^2 + 4y + 4)\,dy \end{aligned}$$

Steps 6 and 7: **Solve and Presentation of Answer:** $\boxed{A = \dfrac{71}{6}.}$

Verify the calculation please!

Example 2.8. ■

2.9. The boundary curves

$$\begin{aligned} y &= \sqrt{1+\sqrt{x}} && 0 \le x \le 4 \\ y &= 0 && \text{the } x\text{-axis} \end{aligned}$$

are written as functions of x; therefore, the natural choice is to use *vertical rules* using x as the variable of integration.

Using Vertical Rules. Vertical rule function is

$$L_v(x) = \sqrt{1+\sqrt{x}} \qquad 0 \le x \le 4$$

The set up of the area integral is

$$A = \int_0^4 \sqrt{1+\sqrt{x}}\, dx. \tag{S-11}$$

This integral is, at our level of play, very difficult to solve.

Using Horizontal Rules. The area integral in (S-11) is nice, but we cannot solve the integral. We *still* want to solve the problem. Let's try *horizontal rules*!

To use horizontal rules, the boundary curves must be written as functions of the ordinate. (See the note on When to Use Horizontal Rules.)

Write boundary curves as functions of y:

$$\begin{aligned} y &= \sqrt{1+\sqrt{x}} \\ y^2 &= 1+\sqrt{x} \\ \sqrt{x} &= y^2 - 1 \\ x &= (y^2-1)^2 \end{aligned}$$

Limits of Integration: Look at the lower and upper extremities of the region. They are easily observed to be $y = 0$ and $y = f(4) = \sqrt{3}$. Thus,

$$A = \int_0^{\sqrt{3}} L_h(y)\,dy$$

where, $L_h(y)$ is the horizontal rule function, yet to be determined.

The Horizontal Rule function: There are a couple of cases to consider: $0 \leq y \leq 1$ and $1 \leq y \leq \sqrt{3}$.

Case 1: $0 \leq y \leq 1$. In this case, $L_h(y) = 4$. (Do you see why?)

Case 2: $1 \leq y \leq \sqrt{3}$.

$$L_h(y) = 4 - (y^2 - 1)^2 \qquad \triangleleft \text{ from (9)}$$

The rule function is then

$$L_h(y) = \begin{cases} 4 & 0 \leq y \leq 1 \\ 4 - (y^2 - 1)^2 & 1 \leq y \leq \sqrt{3} \end{cases}$$

Set up the Area Integral:

$$\begin{aligned} A &= \int_0^{\sqrt{3}} L_h(y)\, dy \\ &= \int_0^1 4\, dy + \int_1^{\sqrt{3}} 4 - (y^2 - 1)^2\, dy \end{aligned}$$

This last area integral, written as the sum of two elementary integral, can be solved.

Calculation of Area Integral:

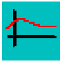
$$
\begin{aligned}
A &= 4y\Big|_0^1 + \int_1^{\sqrt{3}} 3 + 2y^2 - y^4 \, dy \\
&= 4 + 3y + \frac{2}{3}y^3 - \frac{1}{5}y^5\Big|_1^{\sqrt{3}} \\
&= 4 + 3(\sqrt{3} - 1) - \frac{2}{3}(3\sqrt{3} - 1) - \frac{1}{5}(9\sqrt{3} - 1) \\
&= \frac{8}{5} + \frac{16}{5}\sqrt{3}
\end{aligned}
$$

Presentation of Answer: $\boxed{A = \frac{8}{5} + \frac{16}{5}\sqrt{3}.}$

You can see for yourself how changing the orientation of our rules created another integral representing the same area. This new integral was solvable.

Example 2.9. ■

2.10. The figure to the left represents the graph of the two functions



$$y_1 = 4 - x^2 \qquad y_2 = 1 - 2x.$$

Figure S-9

Since we are required to use horizontal rules, this implies that y is the variable of integration. The area integral then looks like

$$A = \int_a^b L(y)\, dy,$$

where $L(y)$ is the horizontal rule function for the region, and the numbers a and b are on the y-axis.

Write Boundary Curves in terms of y: Because we are using horizontal rules, the variable of integration is y. This means that it is necessary to write the boundary curves in terms of y.

$$y = 4 - x^2 \implies x = \pm\sqrt{4 - y}$$

$$y = 1 - 2x \implies x = \frac{1}{2}(1 - y)$$

Let's label these variables for easy reference: Define,

$$x_R = \sqrt{4-y} \qquad x_L = -\sqrt{4-y} \qquad x_2 = \frac{1}{2}(1-y) \tag{S-12}$$

Here, the graph of x_R is the right-hand side of the parabola y_1 and the graph of x_L is the left-hand side of the parabola y_1. (This parabola can be described using a single function of x, but requires two functions of y to describe it.)

Find the Limits of Integration: You'll note from Figure S-9, the lower-most extremity occurs at one of the points of intersection between the two curves y_1 and y_2. While the upper-most extremity of the region occurs at the vertex of the parabola y_1; this is at the point $(\,0, 4\,)$.

The next step is to find where the two curves cross each other.

Points of Intersection: This has been done. In the solution to Example 2.5 it was shown that the two curves intersect when $x - 1$, 3.

The two curves intersect at $x = -1, 3$. We need to find the corresponding ordinates (the y's). Just evaluate either function at each of these two values of x. I'll use the simpler of the two:

$$y_1\big|_{x=-1} = 1 - 2x\big|_{x=-1} = 3 \qquad y_1\big|_{x=3} = 1 - 2x\big|_{x=3} = -5$$

Thus,

$$\text{Points of Intersection: } P(\,-1, 3\,) \text{ and } Q(\,3, -5\,)$$

The limits of integration can now finally be determined. The lower-extremity of the region occurs at $y = -5$ and the upper-extremity occurs at $y = 4$. Thus,

$$A = \int_{-5}^{4} L(y)\,dy. \tag{S-13}$$

where L is the horizontal rule function, still to be determined.

Determining the Horizontal Rule Function: Copy Figure **??** onto a sheet of paper (put in the points of intersection as well). For various values of y, $-4 \le y \le 4$, draw the corresponding horizontal rules. Note that some of them extend on the left form the line $y_2 = 1 - 2x$ over to the parabola $y_1 = 4 - x^2$ on the right. Other horizontal rules extend from the left side of the parabola to the right side of the parabola. In particular,

- If $-5 \le y \le 3$, the corresponding horizontal rule goes from the line on the left to the parabola on the right.
- If $3 \le y \le 4$, the corresponding horizontal rule goes from the left side of the parabola to the right side of the parabola.

Case 1: For any y, $-5 \le y \le 3$, the corresponding horizontal rule extends from the line on the left to the parabola on the right. Thus,

$$L(y) = x_R - x_2 = \sqrt{4 - y} - \frac{1}{2}(1 - y), \qquad \triangleleft \text{ from (S-12)}$$

because the length of a horizontal rule is the abscissa (x) of the right-most point minus the abscissa (x) of the left-most point. (*Note*: abcissa $=$ x-coordinate. See the Calculation of Rules.)

Case 2: For any y, $3 \leq y \leq 4$, the corresponding horizontal rule extends from the left-hand side of the parabola to the right-hand side of the parabola. Thus,

$$L(y) = x_R - x_L = \sqrt{4-y} - (-\sqrt{4-y}), \qquad \triangleleft \text{ from (S-12)}$$

because the length of a horizontal rule is the abscissa of the right-most point minus the abscissa (x) of the left-most point. (*Note*: abcissa = x-coordinate.)

Finally, the rule function can be expressed as,

$$L(y) = \begin{cases} \sqrt{4-y} - \frac{1}{2}(1-y) & -5 \leq y \leq 3 \\ 2\sqrt{4-y} & 3 \leq y \leq 4 \end{cases} \tag{S-14}$$

Set up of Area Integral:

$$\begin{aligned} A &= \int_{-5}^{4} L(y)\,dy && \triangleleft \text{ from (S-13)} \\ &= \int_{-5}^{3} L(y)\,dy + \int_{3}^{4} L(y)\,dy && \triangleleft \text{ Add. Prop.} \\ &= \int_{-5}^{3} \sqrt{4-y} - \frac{1}{2}(1-y)\,dy + \int_{3}^{4} 2\sqrt{4-y}\,dy && \triangleleft \text{ from (S-14)} \end{aligned}$$

Thus,

$$\boxed{A = \int_{-5}^{3} \sqrt{4-y} - \frac{1}{2}(1-y)\,dy + \int_{3}^{4} 2\sqrt{4-y}\,dy.}$$

Calculation of the Area Integral:

$$\begin{aligned} A &= \int_{-5}^{3} \sqrt{4-y} - \frac{1}{2}(1-y)\,dy + \int_{3}^{4} 2\sqrt{4-y}\,dy \\ &= -\frac{2}{3}(4-y)^{3/2} + \frac{1}{4}(1-y)^2\Big|_{-5}^{3} + -\frac{4}{3}(4-y)^{3/2}\Big|_{3}^{4} \\ &= \frac{32}{3} \end{aligned}$$

Presentation of Answer: $\boxed{A = \frac{32}{3}.}$

Example Notes: You'll note the answer computed here is the same one obtained by other methods in Example 2.5—thank goodness.

▪ It doesn't matter, theoretically, whether you use vertical or horizontal rules, the solution to the corresponding area integral will be the correct answer.

■ In terms of total human effort, one rule orientation might be preferred to the other; however, having the ability to apply both methods gives you increased abilities for solving area problems. ■

Example 2.10. ■

3.1. This is a direct application of the formula just developed.

The Set up:

$$V = \int_1^2 [x^2]^2 \, dx \qquad \triangleleft \text{ from } (6)$$

The Calculation:

$$\begin{aligned} V &= \int_1^2 [x^2]^2 \, dx = \int_1^2 x^4 \, dx \\ &= \frac{1}{5} x^5 \Big|_1^2 \\ &= \boxed{\frac{31}{5}} \end{aligned}$$

Example Notes: Notice that it was not necessary to visualize this solid to compute its volume. ■

Example 3.1. ■

3.2. For any x, $0 \leq x \leq \pi$, slice the solid perpendicular to the x-axis at x. We get a circular disk as a cross section. The radius of the this disk is the distance from the center of the circle, which is located at $y = -1$ to the curve $f(x) = 1 + 4x^3$. The radius is given by

Example 3.2. ■

Important Points

The Idea behind the Solution: The solution to this problem is very similar to the one to the *area problem* in Section 7.1 on Integration. Roughly speaking, the idea is to cut narrow paper strips and overlay them onto the target region. (We are sure to cut the heights of our paper strips to approximately conform to the height of the region at that point.) We then pick up our paper strips, measure base times altitude of each to calculate the area of each paper strip. We compute the combined area of all the paper strips. This would represent an approximation of the area of the region. We then *pass to the limit*!

Important Point ■

Why is $L(x) = |f(x) - g(x)|$?

Answer: The distance between two vertically oriented points is the absolute difference of their ordinates. (See the discussion concerning the absolute value function.)

Later, we discuss the rule for the calculation of rules.

Important Point ■

Discussion. The two curves are $x = y^2$ and $x - y = 2$. The first curve is written naturally as a function of y, whereas the second curves is written in equational form. Both curves must be written the same way. Therefore, write

$$x = y^2 \qquad x = y + 2.$$

For convenience and reference, let's name the dependent variables:

$$x_1 = y^2 \qquad x_2 = y + 2$$

This means that we have committed ourselves to *horizontal rules*; consequently, y will be the limit of integration.

$$A = \int_a^b L(y)\,dy,$$

where $L(y)$ is the horizontal rule function, and a and b are the limits of integration—all yet to be determined.

Calculation of the Limits of Integration: To obtain the points of intersection between the two curves, we put $x_1 = x_2$ and solve for y:

$$\begin{aligned} y^2 &= y + 2 \\ y^2 - y - 2 &= 0 \\ (y+1)(y-2) &= 0. \end{aligned}$$

These two curves intersect at $y = -1$ and at $y = 2$.

Update the Integral: We now know that

$$A = \int_{-1}^{2} L(y)\, dy.$$

Calculation of the Rule Function: From equation (9), the easy answer to this question is

$$L(y) = |x_1 - x_2| = |y^2 - (y+2)|.$$

Set Up the Area Integral: $A = \displaystyle\int_{-1}^{2} |y^2 - y - 2|\, dy$. I leave the rest to you.

Important Point ■

Why not just say that x is the variable of integration?

Because, the variable of integration is a *dummy variable*. You can call this variable anything you like! For example, the following integrals are all the same:

$$\int_0^1 x^2\,dx = \int_0^1 y^2\,dy = \int_0^1 t^2\,dt = \int_0^1 w^2\,dw.$$

However, whatever letter you choose to represent the variable of integration, that variable can be thought of as symbol whose values come either from the axis of abscissas (the x-axis) or the axis of ordinates (the y-axis).

This is why I wrote that the variable of integration, "will be on the axis of abscissas."

Important Point ■

Visualize a cookie cutter of arbitrary, yet estetically pleasing shape. Plunge your cookie cutter into a layer of dough of uniform thickness of h. You have just cut out a right cylindrical solid. Important Point ■

The Idea behind the Solution: Think of your solid as a potato. Lay the potato on a counter top and with a shape paring knife begin slicing the potato into thin, almost circular, cross sectional slices. When you are done, you potato is cut into a large number of thin slices.

The volume of the potato is the sum of the volumes of the thin slices (the whole is the sum of its parts). The idea is to estimate the volumes of each potato slice, add up these approximations, represent the total in the form of a *Riemann Sum*, and pass to the limit.

Important Point ■

Define $H(x) = g(x) - f(x) = x^2 + 1 - (\frac{1}{2}x - 1) = x^2 - \frac{1}{2}x + 2$. Note that $H(0) = 2 > 0$ and that

$$H'(x) = 2x - \frac{1}{2} > 0 \text{for } x > \tfrac{1}{4}.$$

This last calculation shows H is increasing on the interval $(\,\frac{1}{4}, \infty\,)$.

What does all this have to do with anything? I'll tell you.

$$\begin{aligned} x > \tfrac{1}{4} \implies & H(x) > H(\tfrac{1}{4}) = 2 > 0 \\ \implies & x^2 + 1 - (\tfrac{1}{2}x - 1) > 0 \\ \implies & x^2 + 1 > \tfrac{1}{2}x - 1. \end{aligned}$$

This shows that, in particular, $0 \leq x \leq 2 \implies g(x) > f(x)$ as required.

This solution illustrates the very powerful differential techniques of creating inequalities. The advantages of this method is that they can be adapted to multivariable variable problems. In these kinds of problems, there may be *no viewable geometry* to utilize. These *sightless methods* are very important.

The same inequality can be also developed by a purely *algebraic* approach—no calculus. This is the method you probably used.

Important Point ■

We first evaluate the integral $\int \sqrt{2-y^2}\,dy$. Indeed, comparing our integral with the formula

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right],$$

we see that $a=\sqrt{2}$, since $a^2=2$. Thus,

$$\int \sqrt{2-y^2}\,dy = \frac{1}{2}\left[y\sqrt{2-y^2}+2\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)\right],$$

Thus,

$$\begin{aligned}\int_0^1 \sqrt{2-y^2}\,dy &= \frac{1}{2}\left[y\sqrt{2-y^2}+2\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)\right]\Bigg|_0^1\\ &= \frac{1}{2}\left[1+2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]\\ &= \frac{1}{2}\left[1+2\left(\frac{\pi}{4}\right)\right]\\ &= \frac{1}{2}+\frac{\pi}{4}\end{aligned}$$

Now, *finally*,

$$
\begin{aligned}
A &= \int_0^1 \sqrt{2-y^2}\,dy - \frac{2}{3} \\
&= \frac{1}{2} + \frac{\pi}{4} - \frac{2}{3} = \boxed{\frac{\pi}{4} - \frac{1}{6}.}
\end{aligned}
$$

Of course, your calculator gives $\sin^{-1}(1/\sqrt{2})$ as a numerical value. Please compare my answer with yours. Important Point ■

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