⇑ toc

13. Presentation of the Theory

In this section we present the proofs of the theorems stated in the main tutorial.

13.1. The Power Rule

Theorem 13.1. (The Power Rule: Junior Grade) *Consider the function $f(x) = x^n$, for some $n \in \mathbb{N}$, the set of natural numbers. Then*

$$f'(x) = nx^{n-1} \qquad x \in \mathbb{R}. \tag{1}$$

Proof. Let $n \geq 1$ be a natural number, and define $f(x) = x^n$. Then,

$$\begin{aligned} f(x+h) &= (x+h)^n \\ &= x^n + nx^{n-1}h + \{\text{all terms have a factor of } h^2\} \end{aligned}$$

Thus, when we form the difference $f(x+h) - f(x)$ the term x^n subtracts away. We obtain

$$f(x+h) - f(x) = nx^{n-1}h + \{\text{all terms have a factor of } h^2\}$$

Now, we divide by h to build the infamous difference quotient.

$$\frac{f(x+h)-f(x)}{h} = nx^{n-1} + \{\text{all terms have a factor of } h\}$$

A factor of h is cancelled from each term. The derivative $f'(x)$ is the limit of this expression as $h \to 0$.

$$f'(x) = nx^{n-1}, \tag{2}$$

the rest of the terms tend to zero since each of them has at least a factor of h in it. But (2) is the desired formula. □

Alternate Proof: Here is an alternate proof using the *Principle of Mathematical Induction*.

13.2. The Algebra of Differentiation

The next few theorems give the proofs of the addition rule, the product rule, and the quotient rule.

Theorem 13.2. (Homogeneity of the Derivative) *Let* $u = f(x)$ *be differentiable at* $x = a$*, then*

$$(cf)'(a) = cf'(a).$$

Proof. Recall the definition of the function (cf): $(cf)(x) = cf(x)$. Then the derivative of the function (cf) is the limit of the difference quotient.

$$\begin{aligned}
(cf)'(a) &= \lim_{h\to 0} \frac{(cf)(a+h) - (cf)(a)}{h} && \triangleleft \text{ difference quotient} \\
&= \lim_{h\to 0} \frac{cf(a+h) - cf(a)}{h} && \triangleleft \text{ defn of } cf \\
&= c \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} && \triangleleft \text{ Homgen. of Limits} \\
&= cf'(a) && \triangleleft \text{ since } f'(a) \text{ exists}
\end{aligned}$$

Thus,

$$\boxed{(cf)'(a) = cf'(a),}$$

which is the desired equation. □

Theorem 13.3. (The Additive Property) *Let $u = f(x)$ and $v = g(x)$ be differentiable at $x = a$. Then*

$$(f+g)'(a) = f'(a) + g'(a). \tag{3}$$

Proof. The equation (3) is verified by the definition. Recall the definition of the sum of two function:

$$(f+g)(x) = f(x) + g(x). \tag{4}$$

Begin by calculating the difference quotient for $f + g$:

$$\begin{aligned}
&\frac{(f+g)(a+h)-(f+g)(a)}{h} && \triangleleft \text{ difference quotient}\\
&\quad = \frac{f(a+h)+g(a+h)-f(a)-g(a)}{h} && \triangleleft \text{ definition of } f+g\\
&\quad = \frac{[f(a+h)-f(a)]+[g(a+h)-g(a)]}{h} && \triangleleft \text{ rearrange}\\
&\quad = \underbrace{\frac{f(a+h)-f(a)}{h}}_{(1)} + \underbrace{\frac{g(a+h)-g(a)}{h}}_{(2)} && \triangleleft \text{ separate fractions}
\end{aligned}$$

Observe that (1) is the difference quotient of the function f, and (2) is the difference quotient for the function g. Now, take the limit as $h \to 0$. The left-hand side of the equation is the difference quotient for the function $f+g$, and the right-hand side is the sum of two difference quotients. Keeping in mind that the limit of a sum is the sum of the limits (provided each limit exits), we obtain:

$$\boxed{(f+g)'(a) = f'(a) + g'(a),}$$

which is the advertised equality.

Theorem 13.4. *Let* $u = f(x)$ *and* $v = g(x)$ *be differentiable at* $x = a$*, then*

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a). \tag{5}$$

Proof. This proof is a little trickier than the previous ones. We begin by constructing the difference quotient and manipulate it appropriately.

$$\begin{aligned}
&\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\
&\qquad = \frac{f(a+h)g(a+h)-f(a+h)g(a)+f(a+h)g(a)-f(a)g(a)}{h}\\
&\qquad = \frac{f(a+h)g(a+h)-f(a+h)g(a)}{h}+\frac{f(a+h)g(a)-f(a)g(a)}{h}\\
&\qquad = f(a+h)\frac{g(a+h)-g(a)}{h}+g(a)\frac{f(a+h)-f(a)}{h}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\
&\qquad = \underbrace{f(a+h)}_{(1)}\underbrace{\frac{g(a+h)-g(a)}{h}}_{(2)\text{ diff. quot.}}+g(a)\underbrace{\frac{f(a+h)-f(a)}{h}}_{(3)\text{ diff. quot.}}. \qquad (6)
\end{aligned}$$

Because f is differentiable as $x = a$, we deduce that f is continuous at $x = a$; this implies that the limit of (1) as $h \to 0$ is $f(a)$. The difference quotient, (2), is a difference quotient for g — the limit of (2) as $h \to 0$ is $g'(a)$ by definition. The other difference quotient, (3), is a difference quotient for f — the limit of (3) as $h \to 0$ is $f'(a)$ by definition.

Now we take the limit as $h \to 0$ of both sides of the equation (6) all the while keeping in mind the observations of the previous paragraph, as well as the various properties of limits. We obtain:

$$\boxed{(fg)'(a) = f(a)g'(a) + g(a)f'(a),}$$

which is the desired computing formula. $\square$

Theorem 13.5. *Let $u = f(x)$ and $v = g(x)$ be differentiable at $x = a$, then*

$$(f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g'(a)]^2}, \tag{7}$$

provided $g(a) \neq 0$.

Proof. This demonstration is the trickiest one of all, but it is similar to the proof of Theorem 13.4.

We begin by calculating and manipulating the difference quotient for the function f/g.

$$\begin{aligned}
&\frac{1}{h}\left[\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}\right] \\
&\qquad = \frac{f(a+h)g(a) - g(a+h)f(a)}{hg(a+h)g(a)} \\
&\qquad = \frac{f(a+h)g(a) - f(a)g(a) + f(a)g(a) - g(a+h)f(a)}{hg(a+h)g(a)} \\
&\qquad = \frac{g(a)(f(a+h) - f(a)) - f(a)(g(a+h) - g(a)}{hg(a+h)g(a)} \\
&\qquad = \frac{\left[g(a)\frac{f(a+h)-f(a)}{h}\right] - f(a)\left[\frac{g(a+h)-g(a)}{h}\right]}{g(a+h)g(a)} \qquad (8)
\end{aligned}$$

You can see the point of the manipulations: to create difference quotients for the purpose of being able to calculate the limit. We want to take the limit as $h \to 0$; the limit of a ratio is the ratio of the limits provided the limit of the denominator is nonzero. Let's check this out:

$$\lim_{h \to 0} g(a+h)g(a) = g(a)g(a) = [g(a)]^2 \neq 0,$$

since we are assuming $g(a) \neq 0$. (Here, g is continuous at $x = a$, since g is differentiable there.)

Finally, take the limit of (8); this quantity is equivalent to the difference quotient of f/g. Thus,

$$(f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2},$$

the desired equality. □

The next theorem simply summarizes the algebraic content of the four previous theorems. This theorem is stated to tie the algebraic structure together.

Theorem 13.6. (The Algebra of Differentiable Functions) *Let f and g be defined in an interval containing $x = a$, and let both f and g be differentiable at $x = a$. Finally let $c \in \mathbb{R}$ be a constant. Then each of the functions are also differentiable at $x = a$ as well:*

$$cf \quad f + g \quad f - g \quad fg \quad \frac{f}{g}, \text{provided } g'(a) \neq 0.$$

Proof. This is the content of Theorem 13.2, Theorem 13.4 and Theorem 13.5.

13.3. The Chain Rule

In this section the chain rule is proved.

Theorem 13.7. (The Chain Rule) *Let $y = f(u)$ and $u = g(x)$ be functions such that f is compatable for composition with g. Suppose g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $f \circ g$ is differentiable at x, and*

$$(f \circ g)'(x) = f'(g(x))g'(x). \tag{9}$$

Proof. Let x be fixed such that g is differentiable at x and f is differentiable at $u = g(x)$. We want to prove $f \circ g$ is differentiable at this same x and (9) holds.

To that end, for any $h \neq 0$, define a variable k by

$$k = g(x+h) - g(x). \tag{10}$$

Note that

$$g(x+h) = g(x) + k = u + k \tag{11}$$

The difference quotient for the function $f \circ g$ is

$$\frac{f(g(x+h)) - f(g(x))}{h}, \tag{12}$$

we want to argue that the limit of this expression as $h \longrightarrow 0$ exists and is equal to the right-hand side of (9).

To clearly demonstrate our stated goal, it is necessary to define a new function

$$F(k) = \begin{cases} \dfrac{f(u+k) - f(u)}{k} & k \neq 0 \\ f'(u) & k = 0 \end{cases}$$

where in the definition of F we have used the letters u and k. Think of u as $u = g(x)$, but for right now, think of k as just a mathematical variable. Later, we will put $k = g(x+h) - g(x)$. The properties of F are that F is continuous at k since

$$\lim_{k \to 0} F(k) = \lim_{k \to 0} \frac{f(u+k) - f(u)}{k} = f'(u) = F(0) \tag{13}$$

Also, it is important to note that

$$kF(k) = \begin{cases} f(u+k) - f(u) & k \neq 0 \\ 0 & k = 0 \end{cases}$$

Or, better yet, we can see that

$$F(k)k = f(u+k) - f(u) \qquad \text{for all } k \tag{14}$$

since when $k = 0$, the right-hand expression reduces to 0.

Now, returning to the difference quotient (12) we get

$$
\begin{aligned}
\frac{f(g(x+h) - f(g(x))}{h} &= \frac{f(u+k) - f(u)}{h} && \triangleleft \text{ by (11)} \\
&= \frac{F(k)k}{h} && \triangleleft \text{ by (14)} \\
&= F(k)\frac{k}{h} \\
&= F(k)\frac{g(x+h) - g(x)}{h} && \triangleleft \text{ from (10)}
\end{aligned}
$$

Now, as $h \longrightarrow 0$, since g is differentiable at x,

$$\lim_{h \longrightarrow 0} k = \lim_{h \longrightarrow 0} g(x+h) - g(x) = 0$$

it is necessarily continuous at x as well. By the **Composite Limit Theorem** we have

$$\lim_{h \longrightarrow 0} F(k) = F(0).$$

But, $F(0) = f'(u)$. Finally, we make the calculation:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(g(x+h) - f(g(x))}{h} &= \lim_{h \to 0} \left[F(k) \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} F(k) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= F(0) g'(x) \\
&= f'(u) g'(x) \\
&= f'(g(x)) g'(x)
\end{aligned}$$

This completes the proof. □

13.4. The Trigonometric Functions

In this section we present proofs of some of the differentiation results concerning trigonometric functions.

Theorem 13.8. *The derivatives of the trigonometric functions are given by the following sets of formulas:*

(1) $\dfrac{d}{dx}\sin(x) = \cos(x)$ (2) $\dfrac{d}{dx}\cos(x) = -\sin(x)$

(3) $\dfrac{d}{dx}\tan(x) = \sec^2(x)$ (4) $\dfrac{d}{dx}\cot(x) = -\csc^2(x)$

(5) $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$ (6) $\dfrac{d}{dx}\csc(x) = \csc(x)\cot(x)$

Proof. We calculate each of these in turn. The derivatives of the sine and cosine function are done by the definition; the other four functions are rational functions of sine and cosine, they can be computed by the quotient rule.

The Derivative of the Sine Function: We compute the derivative directly from the definition. To do this, we need to use the addition formula for the sine function: Let A and B be numbers, then

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A). \tag{15}$$

The difference quotient is

$$\frac{\sin(x+h)-\sin(x)}{h} = \frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$
$$= \sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}$$

The derivative is the limit of the difference quotient as h tends to zero.

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \\ &= \sin(x)\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{(1)} + \cos(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{(2)}\end{aligned} \tag{16}$$

Recall from our earlier work that

$$(1)\text{:}\ \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0 \qquad (2)\text{:}\ \lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

Substituting these back into (16) we obtain:

$$\boxed{\frac{d}{dx}\sin(x) = \cos(x).} \tag{17}$$

which is the advertised formula.

The Derivative of the Cosine Function: We compute this derivative using the definition as well. This computation can be made in a manner virtually identical to that of the sine function: We would use the addition formula for cosine function

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B), \tag{18}$$

as well as the two limit results used in the previous demonstration. Rather than mimicking something you have already seen, let's have a different approach.

It follows from (15) that

$$\cos(x) = \sin(\pi/2 - x) = -\sin(x - \pi/2).$$

The last equality also follows from (15) (Verify?), but it is just the much celebrated property of the sine function — that of being an *odd function*: $\sin(-A) = -\sin(A)$, for any number $A \in \mathbb{R}$.

Now invoke the definition of derivative:

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} \\
&= -\lim_{h\to 0}\frac{\sin(x-\pi/2+h)-\sin(x-\pi/2)}{h} \\
&= -\frac{d}{dt}\sin(t)\bigg|_{t=x-\pi/2} && \triangleleft\ (17) \\
&= -\cos(t)\big|_{t=x-\pi/2} \\
&= -\cos(x-\pi/2) = -\sin(x) && \triangleleft\ \text{by } (18)
\end{aligned} \tag{19}$$

You should verify in your own mind the validity of equality (19). But the limit in the (19) is also the same difference quotient you would obtain if you were trying to calculate the derivative of $\sin(t)$ at the particular value of $t = x - \pi/2$ — here I have introduced a dummy variable for clarity (or confusion). Study the argument carefully to assure your understanding.

Thus we have argued

$$\boxed{\frac{d}{dx}\cos(x) = -\sin(x).}$$

The Derivative of the Tangent Function: The tangent function is the ratio of the sine function and the cosine function — just use the quotient rule.

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos(x)\dfrac{d}{dx}\sin(x) - \sin(x)\dfrac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}
$$

Thus, we have verified that

$$\boxed{\frac{d}{dx}\tan(x) = \sec^2(x).} \tag{20}$$

The Derivative of the Cotangent Function: The cotangent function is the ratio of the cosine function by the sine function:

$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

The derivative can be calculated in exactly the same way as the tangent function. Just to change things up a little, and to keep you on your toes, let's write,

$$\cot(x) = \frac{1}{\tan(x)};$$

therefore,

$$\begin{aligned}\frac{d}{dx}\cot(x) &= \frac{d}{dx}\frac{1}{\tan(x)} \\ &= \frac{0-(1)\sec^2(x)}{\tan^2(x)} && \triangleleft \text{ quot. rule \& (20)} \\ &= -\frac{\sec^2(x)}{\tan^2(x)} \\ &= -\csc^2(x)\end{aligned}$$

You must verify the last step for yourself — verify, verify, verify everything!

Thus, we have shown,

$$\boxed{\frac{d}{dx}\cot(x) = -\csc^2(x).}$$

The Derivative of the Secant Function: Now for the secant function.

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\frac{1}{\cos(x)} \\
&= \frac{0 - (1)[-\sin(x)}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \sec(x)\tan(x)
\end{aligned}$$

Thus we have shown,

$$\boxed{\frac{d}{dx}\sec(x) = \sec(x)\tan(x).}$$

The Derivative of the Cosecant Function: This can be computed in the same way as the case of the secant function:

$$\begin{aligned}\frac{d}{dx}\csc(x) &= \frac{d}{dx}\frac{1}{\sin(x)} \\ &= \frac{0-(1)[\cos(x)]}{\sin^2(x)} \\ &= -\frac{\cos(x)}{\sin^2(x)} \\ &= -\csc(x)\cot(x).\end{aligned}$$

The last step should be, of course, verified.

Thus,

$$\boxed{\frac{d}{dx}\csc(x) = -\csc(x)\cot(x).}$$

This completes the proof of Theorem 13.8 □

13.5. The Mean Value Theory

This section is devoted to developing the theory leading up to and including the MEAN VALUE THEOREM.

Theorem 13.9. (Fermat's Theorem) *Let f be a function defined on an interval I. Suppose $c \in I$ is a local extremum such that c is not an endpoint of I and $f'(c)$ exits, then $f'(c) = 0$.*

Proof. We can assume, without loss of generality, that c is a local minimum. Then, there exists an interval J such that for all $x \in J \cap I$

$$f(x) \geq f(c). \tag{21}$$

Because c is not an endpoint of the interval I, we can show that $f'(c) = 0$ by calculating the left-hand derivative and the right-hand derivative of f at c.

Left-hand Derivative: Recall: $f'_{-}(c) = \lim_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h}$.

Towards that end: Let $h < 0$ be so small that $c + h \in J$, then from (21),

$$f(c+h) \geq f(c)$$

or,

$$f(c+h) - f(c) \geq 0.$$

Now, as we are assuming $h < 0$, we see that

$$\frac{f(c+h) - f(c)}{h} \leq 0.$$

Therefore,

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \leq 0$$

thus,

$$\boxed{f'_-(c) \leq 0} \tag{22}$$

Right-hand Derivative: $f'_+(c) = \lim\limits_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h}$.

Let $h > 0$ be so small that $c + h \in J$, then from (21),

$$f(c+h) \geq f(c)$$

or,

$$f(c+h) - f(c) \geq 0.$$

Now, as we are assuming $h > 0$, we see that

$$\frac{f(c+h) - f(c)}{h} \geq 0.$$

Therefore,

$$f'_{-}(c) = \lim_{h \to 0^{-}} \frac{f(c+h) - f(c)}{h} \geq 0$$

thus,

$$\boxed{f'_{+}(c) \geq 0} \tag{23}$$

Conclusions: We have shown (22) and (23). What does this mean? Recall, that since $f'(c)$ exists, this implies

$$f'(c) = f'_{-}(c) = f'_{+}(c).$$

We have, then by (22) and (23), that

$$0 \leq f'_+(c) = f'(c) \text{ and } 0 \geq f'_-(c) = f'(c),$$

or, more simply,

$$f'(c) \geq 0 \text{ and } f'(c) \leq 0.$$

These two inequalities mean $f'(c) = 0$, which is the advertised result. □

Theorem 13.10. (Rolle's Theorem) *Let f be a function be continuous on the interval $[\,a, b\,]$ and differentiable on the interval $(\,a, b\,)$ such that*

$$f(a) = f(b)$$

Then there exists a number $c \in (\,a, b\,)$ such that $f'(c) = 0$.

Proof. By the Extreme Value Theorem, there is a number $x_{\min} \in [\,a, b\,]$ and a number $x_{\max} \in [\,a, b\,]$ such that

$$f(x_{\min}) = \min_{a \leq x \leq b} f(x) \qquad f(x_{\max}) = \max_{a \leq x \leq b} f(x),$$

that is, f attains its absolute minimum and absolute maximum at $x_{\min}$ and $x_{\max}$, respectively.

Now, if f is a constant function over the interval $[\,a, b\,]$, the conclusion of the theorem is obvious. Assume, therefore, that f is non constant; this implies that either f takes on a value larger than $f(a)$ or f takes on a value smaller than $f(a)$. For the purpose of argument, we make a further assumption that the former is true, i.e. f takes on at least one value greater than $f(a)$.

Based on the assumption that f takes on a value larger than $f(a)$, we conclude $f(x_{\max}) > f(a)$, since $f(x_{\max})$ is the largest value of f; furthermore, it must be true that $a < x_{\max} < b$, since this maximum value cannot be attained at a, nor can it be attained at b, since $f(a) = f(b)$ (remember?).

Define the c referenced in the conclusion of the theorem as

$$c = x_{\max}.$$

Finally, we need to argue that $f'(c) = 0$, but this follows from FERMAT'S THEOREM: f indeed has a local extrema at c (since it is in fact an absolute extrema), and as we have deduced, c is not an endpoint. Since f is differentiable on the interval (a, b) and $c \in (a, b)$, $f'(c)$ exits. We have just argued that the hypothesis of FERMAT'S THEOREM is true for this situation and are justified in concluding $f'(c) = 0$. $\square$

Theorem 13.11. (The Mean Value Theorem) *Let f be a function continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Then there exists a number $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{25}$$

or,

$$f(b) - f(a) = f'(c)(b - a) \tag{26}$$

or even,

$$f(b) = f(a) + f'(c)(b - a) \tag{27}$$

Proof. Let $L(x)$ the function whose graph is a straight line passing through the two points

$$(\,a, f(a)\,) \text{ and } (\,b, f(b)\,).$$

The exact functional form can be worked out if desired, but it is not needed. L is a straight line whose slope is

$$m = \frac{f(b) - f(a)}{b - a}. \tag{28}$$

Recall though that the derivative of a straight line is its slope; therefore,

$$L'(x) = \frac{f(b) - f(a)}{b - a}.$$

Now, define a new function g by

$$g(x) = f(x) - L(x). \tag{29}$$

Note that,

$$g(a) = 0 = g(b)$$

We call on ROLLE'S THEOREM to conclude that there exists a number $c \in (a, b)$ such that

$$g'(c) = 0. \tag{30}$$

But, from (29)

$$g'(c) = f'(c) - L'(c).$$

Taking (28) and (30) into consideration we conclude

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}, \tag{31}$$

and so,

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which is the desired equation. $\square$

Theorem 13.12. (The First Derivative Test) *Let f be a function and c a critical point of f.*

(1) Test for Local Maximum. *Suppose f' is positive to the left of c and negative to the right of c, then c is a local maximum.*
(2) Test for Local Minimum. *Suppose f' is negative to the left of c and positive to the right of c, then c is a local minimum.*
(3) Test for a Saddle Point. *Suppose f' does not change signs at c, then c is a saddle point.*

Important Points

The theorem has already been proved for the case $n = 1$.

Now assume the power rule is true for the natural number n. We want to prove it true for the natural number $n+1$. Indeed, let $f(x) = x^{n+1}$, then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{n+1} - x^{n+1}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+h)^n - x^{n+1}}{h} \\
&= \lim_{h\to 0} \frac{x(x+h)^n + h(x+h)^n - x^{n+1}}{h} \\
&= \lim_{h\to 0} \frac{x[(x+h)^n - x^n] + h(x+h)^n}{h} \\
&= \lim_{h\to 0} \left[x\frac{(x+h)^n - x^n}{h} + (x+h)^n \right] \\
&= x(nx^{n-1}) + x^n \\
&= nx^n + x^n
\end{aligned}$$

$$= (n+1)x^n.$$

Thus we have shown that if $f(x) = x^{n+1}$, then $f'(x) = (n+1)x^n$. By the *Principle of Mathematical Induction*, we have shown

$$\frac{dx^n}{dx} = nx^{n-1} \qquad \forall n \in \mathbb{N}.$$

This proves the theorem. $\square$

Important Point ■