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toc

8. The Rule of Chains

Addition, subtraction, multiplication, and division are some of the ways we use to build up new functions from old, and we have seen how the operation of differentiation interacts with these basic arithmetic processes. There is, as you know, one other way of combining functions: *composition*.

A thorough study of composition of functions has already been taken up; included in that discussion were the topics of composition, the mechanics of composition, and the very important notion of *uncomposing* or *decomposing* a function.

In this section, we study how differentiation interacts with composition. The formal statement of this relationship is the *Chain Rule*, and is stated immediately below.

As it turns out, the most important skill associated with a consistently correct use of the *Chain Rule* is the ability to *realize* that a given function is, in fact, the composition of other functions, *and* the ability to *identify* these functions. This *realiaztion* and *identification* is roughly the process of uncomposing mentioned and referenced above.

Theorem 8.1. (The Chain Rule) *Let $y = f(u)$ and $u = g(x)$ be functions such that f is compatable for composition with g. Suppose g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $f \circ g$ is differentiable at x, and*

$$(f \circ g)'(x) = f'(g(x))g'(x). \tag{1}$$

Proof.

Theorem Notes: The Chain Rule can be very mystifying when you see it and use it the first time. Hopefully, this article will clear this up for you.

- The Chain Rule allows us to differentiate a more complicated function by multiplying together the derivatives of the functions used to compose the parent function.
- In the theorem, I have conveniently labeled the functions and their variables in such a way as to suggest the composition: $y = f(u)$

and $u = g(x)$. Needless to say, generally, functions are not labeled in the most pleasing manner.

■ Theorems are often stated in a very clean, precise and pedagogical manner, or, as often happens, not stated in a way they are actually used in practice. In the case of this theorem, usually, you are given a function, say $F(x) = (x^2+1)^{10}$, to differentiate, the problem becomes to uncompose the function. You must realize that F is the composition of two functions: $f(x) = x^{10}$ and $g(x) = x^2 + 1$. (Check for yourself that $F(x) = f(g(x))$.) Then you apply the theorem as stated. What is not mentioned in the theorem is that **you** must uncompose your function to apply the theorem. These *decomposition methods* were covered in a paragraph on uncomposing functions. ■

EXAMPLE 8.1. Find the derivative of $F(x) = (x^2+1)^{10}$ using the Chain Rule.

Listed below is a chain of thoughts that are necessary to be successful at applying the *Chain Rule*.

Chain Rule: The Procedure.

Step: 1 Recognize and realize that the given function is a composition of two (or more!) other functions.
Step: 2 Decompose the given function.
Step: 3 Apply the Chain Rule

Here's another example, this one numerical.

EXAMPLE 8.2. Calculate $F'(2)$, where $F(x) = (3x^3 - x)^7$.

Actually, both of the past examples followed same pattern. Presently, we will identify this pattern; in this way, we don't have to go through such painful analysis every time.

Review the method of solution of EXAMPLE 8.1 and EXAMPLE 8.2, then solve the following exercise. Solve it completely before looking at the solution.

EXERCISE 8.1. Find the derivative of $F(x) = (3x^4 + 5x)^{1/2}$ using the Chain Rule.

The *Chain Rule* can be applied in many situations. In the above examples, the *Chain Rule* is utilized to evaluate the derivative of specific functions. In the next example, the *Chain Rule* is used to differentiate the composition of an abstract function with a specific function. Confused? Read on.

EXAMPLE 8.3. Let f be a differentiable function, and define a new function by

$$F(x) = f(x^3).$$

Calculate $F'(x)$ using the chain rule.

These kinds of problem types are encountered in *differential equations* and other higher mathematical disciplines.

Study the reasoning of the previous example, then tackle the following problem.

EXERCISE 8.2. Let f be a differentiable function, and define a new function by

$$F(x) = f(1/x^4).$$

Calculate $F'(x)$ using the chain rule.

EXERCISE 8.3. Let g be a differentiable function, and define

$$F(x) = [g(x)]^6.$$

Calculate $F'(x)$ using the Chain Rule.

8.1. Chaining with Leibniz

Let's look at the Chain Rule from the point view of the Leibniz notation. (Can notation have a point of view?)

Let $y = f(u)$ and $u = g(x)$ be compatible for composition. When we compose, this establishes the y as a function of the x variable. If y is considered a function of x, then the functional relationship is given by $y = (f \circ g)(x) = f(g(x))$. The Leibniz notation for that is

$$(f \circ g)'(x) = \frac{dy}{dx} = \text{the derivative of the } y\text{-variable when it is considered a function of } x \text{, or with respect to } x.$$

But y is naturally a function of u since $y = f(u)$; thus

$$f'(u) = \frac{dy}{du} = \text{the derivative of the } y\text{-variable when it is considered a function of } u\text{, or with respect to } u$$

But u is naturally a function of x since $u = g(x)$; so,

$$g'(x) = \frac{du}{dx} = \text{the derivative of the } u\text{-variable when it is considered a function of } x\text{, or with respect to } x$$

See the discussion on the Leibniz Notation for a refresher course.

The Chain Rule can now be translated. The Chain Rule

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

becomes

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \tag{2}$$

Let's make this formula into a big deal:

The Chain Rule: Let $y = f(u)$ and $u = g(x)$ be differentiable and compatible for composition, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The next example illustrates how the Leibniz form of the chain rule is used.

Example 8.4. Calculate: $\dfrac{d}{dx}(5x^4 - 12x^2)^3$.

Use the Leibniz Notation to solve the following problem.

Exercise 8.4. Calculate: $\dfrac{d}{dx}(3x^3 - 6x)^{1/2}$.

8.2. The Power Rule Revisited

As was advertised earlier, we don't to go through this painful pulling of teeth. All example thus far followed the same pattern. Let us state the *Generalized Chain Rule*!

Let $u = f(x)$ be a differentiable function of x and let $r \in \mathbb{Q}$, the set of all rational numbers. Consider the problem of differentiating the function $[f(x)]^r$. We use chain rule techniques:

$$y = u^r \text{ where } u = f(x).$$

We have our decomposed function setup, and we want to calculate dy/dx. This is a job for the Chain Rule!

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{chain it} \\ &= \frac{d\,u^r}{du}\frac{du}{dx} \\ &= ru^{r-1}\frac{du}{dx} && \text{Power Rule} \qquad (3) \end{aligned}$$

Now, substituting for $u = f(x)$ we obtain the formula,

$$\frac{d}{dx}[f(x)]^r = r[f(x)]^{r-1} f'(x).$$

This is a very nice formula, but I prefer the more laconic version (3)

Generalized Power Rule:
Let u be a function of x and $r \in \mathbb{Q}$, the set of rational numbers, then

$$\blacksquare \qquad \frac{d\,u^r}{dx} = r u^{r-1} \frac{du}{dx}.$$

Utilizing the Generalized Power Rule, we can differentiate complex functions with great ease.

EXAMPLE 8.5. (Skill Level 1) Calculate $\dfrac{d}{dx}(1 - 3x^3)^{10}$.

EXAMPLE 8.6. Calculate $\dfrac{d}{ds}\dfrac{1}{(s^4 - s + 1)^{3/4}}$.

EXERCISE 8.5. Calculate $\dfrac{d}{dx}(4x^2 + 1)^{23}$.

EXERCISE 8.6. Calculate $\dfrac{d}{dx}\left(\dfrac{x}{1+x}\right)^5$.

EXAMPLE 8.7. Calculate $\dfrac{d}{dx}x^3\sqrt{1 + 3x^2}$.

EXERCISE 8.7. Calculate $\dfrac{d}{dx}x^4(2x + 1)^{3/2}$.

Let's combine the Power Rule with the quotient rule.

EXAMPLE 8.8. Calculate $\dfrac{d}{dw}\dfrac{w\sqrt{w}}{(3w^3 + 1)^6}$.

EXERCISE 8.8. Calculate $\dfrac{d}{ds}\dfrac{(s+1)^3}{(2s + 1)^5}$.

9. The Trigonometric Functions

In this section, we tackle the problem of differentiating the Trigonometric functions.

9.1. Development of Trig Formulas

Let $f(x) = \sin(x)$. If we wanted to calculate the derivative of f we first setup the Difference Quotient:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin(x)}{h} \\
&= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \qquad (1) \\
&= \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
&= \sin(x)\frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h} \qquad (2)
\end{aligned}$$

Where we have used the *additive formula* for $\sin(x)$ in (1) in the above manipulations. Recall,

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h).$$

Now, we take the limit of (2) as h goes to 0.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}\right) \\ &= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \end{aligned} \tag{3}$$

The equation (3) now makes is it clear the nub of the problem: We need to calculate two limits

$$\lim_{h\to 0}\frac{\sin(h)}{h} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} \tag{4}$$

But these two limits have been already been studied. We have shown that

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0 \tag{5}$$

Now we are ready to calculate the derivative of the function $\sin(x)$. It's been awhile, but let's continue our derivative calculations. Recall, $f(x) = \sin(x)$,

$$\begin{aligned} f'(x) &= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} && \text{from (3)} \\ &= \sin(x)(0) + \cos(x)(1) && \text{from (5)} \\ &= \cos(x) \end{aligned}$$

Thus, we have shown that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$. Using the Leibniz notation we obtain

$$\boxed{\frac{d}{dx} \sin(x) = \cos(x) \qquad x \in \mathbb{R}.} \tag{6}$$

Let's declare victory over the problem of differentiating the trigonometric functions by stating complete results.

Theorem 9.1. *The derivatives of the trigonometric functions are given by the following sets of formulas:*

(1) $\dfrac{d}{dx}\sin(x) = \cos(x)$ (2) $\dfrac{d}{dx}\cos(x) = -\sin(x)$

(3) $\dfrac{d}{dx}\tan(x) = \sec^2(x)$ (4) $\dfrac{d}{dx}\cot(x) = -\csc^2(x)$

(5) $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$ (6) $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Proof.

Theorem Notes: I have arranged the trigonometric functions into three sets of two formulas each. If you stare at the three sets, you might see a pattern. Observing this pattern enables you to reduce from 6 the number of formulas to 3 — that's a 50% decrease in the number of allocated brain cells!

- Each trig function has a "co"-function. The cofunction of $\sin(x)$ is the $\cos(x)$. (The cofunction of $\cos(x)$ is $\sin(x)$.) The cofunction of $\tan(x)$ is $\cot(x)$, and the cofunction of $\sec(x)$ is $\csc(x)$.

■ Formula (2) is obtained by "co-ing" formula (1), and prefixing a negative sign. Formula (4) is obtained by "co-ing" formula (3), and prefixing a negative sign. Formula (6) is obtained by "co-ing" formula (5) and prefixing a negative sign. That's the pattern.

■ Therefore, it suffices to know formulas (1), (3), and (5), and know the "co-ing" trick. ■

EXERCISE 9.1. Calculate $\dfrac{d}{dx}[3\cos(x)]$ and $\dfrac{d}{dt}[-5\sec(t)]$. You may want to review the trigonometric formulas

EXAMPLE 9.1. Find the slope of the line tangent to the graph of $y = \sin(x)$ at $x = \pi/3$. Find the equation of the line tangent to the graph at $x = \pi/3$.

Now, here's one for you.

EXERCISE 9.2. Find the slope of the line tangent to the graph of $y = \tan(x)$ at $x = \pi/3$. Find the equation of the line tangent to the graph at $x = \pi/3$.

Exercise 9.3. Using $\dfrac{d}{dx}\sin(x) = \cos(x)$ and $\dfrac{d}{dx}\cos(x) = -\cos(x)$ *only*, calculate $\dfrac{d}{dx}\tan(x)$.

Now that we have the trig functions, we can combine them with other functions through addition, multiplication, and division.

Example 9.2. Calculate $\dfrac{d}{dx}x^3\sin(x)$.

To test your understanding of the *Chain Rule*, consider the following exercise.

Exercise 9.4. Find the derivative of $F(x) = \sin(6x^3)$ using the Chain Rule.

9.2. The Chain Rule Revisited

Let's apply the Chain Rule to a very common situation.

Suppose we have a function $y = \sin(x^2)$, and we want to calculate $\frac{dy}{dx}$. Set up the composition: Let $y = \sin(u)$, and $u = x^2$, then by the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{d}{du}\sin(u)\frac{d\,x^2}{dx} \\ &= \cos(u)(2x) \\ &= \cos(x^2)(2x)\end{aligned}$$

Thus, we have shown

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)(2x) = 2x\cos(x^2).$$

Using this same reasoning process, we can take the basic trig formulas, and generalize them to allow arbitrary differentiable arguments.

Let u be a differentiable function of x, then

$$(1)\,\frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx} \qquad (2)\,\frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx}$$
$$(3)\,\frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx} \qquad (4)\,\frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}$$
$$(5)\,\frac{d}{dx}\sec(u) = \sec(x)\tan(u)\frac{du}{dx} \qquad (6)\,\frac{d}{dx}\csc(u) = -\csc(x)\cot(u)\frac{du}{dx}$$

Example 9.3. (Skill Level 0) Calculate $\dfrac{d}{dx}\cos(3x^4)$.

A piece of good advice: Do what a formula says to do, no more, no less, and you can't go wrong. Have the courage to do the obvious.

Exercise 9.5. Calculate $\dfrac{d}{dx}\tan(\sin(x))$. (Hint: Follow the advice above)

Having gained experience, try this one on for size.

Exercise 9.6. (Chaining Observed) Calculate $\dfrac{d}{dx}\sin(\tan^2(3x^3))$.

Summary: Let's catalog our new formulas *one more time*!

Trigonometric Differentiation Formulas:

$$(1)\ \frac{d}{dx}\sin(u) = \cos(u)\frac{du}{dx} \qquad (3)\ \frac{d}{dx}\tan(u) = \sec^2(u)\frac{du}{dx}$$

$$(2)\ \frac{d}{dx}\cos(u) = -\sin(u)\frac{du}{dx} \qquad (4)\ \frac{d}{dx}\cot(u) = -\csc^2(u)\frac{du}{dx}$$

$$(5)\ \frac{d}{dx}\sec(u) = \sec(x)\tan(u)\frac{du}{dx}$$

$$(6)\ \frac{d}{dx}\csc(u) = -\csc(x)\cot(u)\frac{du}{dx}$$

10. Higher Order Derivatives

Given a differentiable function f, we have seen that f' is also a function. We have also seen the Prime Notation convention: if a function has a name, the name of its derivative is obtained by taking the parent function and post-fixing a prime ($'$).

Because f' is itself a function, we can attempt differentiating it. The derivative of f' is denoted by f'' and is called the *second derivative* of f.

Example 10.1. For $f(x) = x^5$ calculate the first and second derivatives of f.

Similarly, f'' is a function as well, so we may differentiate it. The derivative of f'' is denoted by f''' and is called the *third derivative* of f ... and so on.

Notation. The prime notation has a weakness. Write symbolically, the tenth derivative of f. *Answer*: f''''''''''. Hopefully, I counted my primes correctly. At some point, the prime notation becomes a burden, so we switch over to another. The fourth derivative of f is denoted by $f^{(4)}$, the fifth derivative is denoted by $f^{(5)}$. In general, if $n \in \mathbb{N}$, then the n^{th} derivative is denoted

$$f^{(n)} = \text{the } n^{\text{th}} \text{ derivative of } f.$$

The integer n in the derivative $f^{(n)}$ is called the*order of the derivative*.

Example 10.2. (Example 10.1Continued) For $f(x) = x^5$, calculate all derivatives of f.

When the function has an anonymous name, we use the prime notation as well.

$$
\begin{aligned}
y &= \sin(2x) \qquad (1)\\
y' &= 2\cos(2x)\\
y'' &= -4\sin(2x)\\
y''' &= -8\cos(2x)\\
y^{(4)} &= 16\sin(2x)\\
y^{(5)} &= 32\cos(2x)\\
y^{(6)} &= -64\sin(2x).
\end{aligned}
$$

In the case of the sine function, we get nonzero derivatives of all orders.

EXERCISE 10.1. Verify that for $y = \sin(2x)$, a general formula for $y^{(n)}$ is

$$
y^{(n)} = 2^n \sin(2x + n\pi/2) \qquad n = 1, 2, 3, 4, \ldots.
$$

The Leibniz notation for higher order derivatives may seem a little strange at first. If $y = f(x)$, then

$$\begin{aligned} y' &= \frac{dy}{dx} \\ y'' &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \\ y''' &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \\ y^{(4)} &= \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4}. \end{aligned}$$

In general, for $n \in \mathbb{N}$, the n^{th} derivative with respect to x is given by

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}},$$

and is calculated by differentiating the $(n-1)^{\text{st}}$-derivative.

For example, if $y = x^4$, then

$$\begin{aligned}
\frac{dy}{dx} &= 4x^3 \\
\frac{d^2y}{dx^2} &= 12x^2 \\
\frac{d^3y}{dx^3} &= 24x \\
\frac{d^4y}{dx^4} &= 24 \\
\frac{d^5y}{dx^5} &= 0
\end{aligned}$$

Note that the derivative of each line is the next line.

EXERCISE 10.2. Let $y = \dfrac{x}{2x+1}$, calculate y' ,y'', and y'''.

11. Implicit Differentiation

The title of this section is *implicit differentiation*. What is *explicit differentiation*? Read on.

11.1. Statement of the Problem

Consider the following equation: $x^2 + y^2 = 1$. As you know, the graph of this equation is a circle of radius $r = 1$ with center at the origin. This curve does *not* define y as a function of x as it fails the vertical line test.

Despite not being a function, one feels that $x^2 + y^2 = 1$ has tangent lines to its graph. How do we calculate dy/dx? The techniques developed up to this point requires that y be an *explicit* function of x. Then we can differentiate — maybe.

Solving for y in $x^2 + y^2 = 1$, we get

$$y = \pm\sqrt{1 - x^2} \qquad |x| \leq 1. \tag{1}$$

The presence of the $\pm$ again suggests that y is *not* a function of x: Each x has 2 associated y-values.

Another point of view of (1) is that the circle can be described by a single equation, $x^2 + y^2 = 1$, but requires *two* functions to describe its curve; namely

$$y = \sqrt{1 - x^2} \qquad |x| \leq 1 \tag{2}$$

and

$$y = -\sqrt{1 - x^2} \qquad |x| \leq 1 \tag{3}$$

The first function describes the *upper semi-circle* and the second function describes the *lower semi-circle*.

Therefore, if we wanted the slope of a line that is tangent to the upper semi-circle, we would differentiate (2) to get

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1 - x^2}} \qquad |x| < 1 \tag{4}$$

Similarly, if we wanted the slope of a line that is tangent to the lower semi-circle, we would differentiate (3) to get

$$\frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}} \qquad |x| < 1 \tag{5}$$

For example, the point $(x, y) = (1/2, \sqrt{3}/2)$ lies on the unit circle $x^2 + y^2 = 1$. Find the slope of the tangent line to this point. Well, the given point lies on the upper semi-circle, so we would use the formula (4):

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(x,y)=(1/2,\sqrt{3}/2)} &= -\frac{1/2}{\sqrt{1 - (1/2)^2}} \\ &= -\frac{1}{\sqrt{3}}. \end{aligned} \tag{6}$$

Or, we may be interested in a point on the lower semi-circle: $(x, y) = (1/2, -\sqrt{3}/2)$. In this case, we use the formula (5),

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1/2,\sqrt{3}/2)} = \frac{1/2}{\sqrt{1-(1/2)^2}} = \frac{1}{\sqrt{3}}. \tag{7}$$

In this case, all calculations are the same except for the sign.

As you can see from this extended example, this is the kind of procedure you would have to do if the curve is given in equational form: (1) solve for y in terms of x – you may get several solutions, each defining y as a function of x; (2) differentiate the functional relation of interest in order to obtain dy/dx.

This method may be rather drawn out, or even impossible. Some equations are so complicated that you cannot algebraically solve of y in terms of x. In the equation,

$$x^{10} - 2xy - y^{10} = 0.$$

it is not possible to algebraically solve for y in terms of x. Yet, the equation defines some curve and this curve has tangent lines. If our goal was to calculate dy/dx we could not do so because we cannot write y in the form of a function of x!

Therefore, another more powerful, more flexible method is needed. This is the method of *implicit differentiation*. It is explained in the next section.

11.2. The Technique Explained

Let's state, in clear terms, the problem.

The Given. Given an equation

$$F(x, y) = c, \tag{8}$$

that is, we are given some equation ($F(x, y) = c$) involving the variables x and y.

The Problem. Find $\dfrac{dy}{dx}$.

The Procedure.

1. Treat y as if it were an explicit function of x.
2. Differentiate both sides of (8) with respect to x. Use the various differentiation formulas wherever applicable. When finished, we will have an equation involving x, y, and dy/dx. Symbolically,

$$G(x, y, \frac{dy}{dx}) = 0 \tag{9}$$

3. Solve the equation (9) for dy/dx.

Let's implement this procedure in a series of examples.

Example 11.1. Find $\dfrac{dy}{dx}$, for $x^2 + y^2 = 1$ using *implicit differentiation*.

Next up is an example in which it is impossible to solve for y; consequently, implicit differentiation is the only way we have to differentiate this curve.

Example 11.2. Calculate $\dfrac{dy}{dx}$, where $x^{10} - 2xy - y^{10} = 0$.

EXAMPLE 11.3. Consider the equation $x \sin(xy) = 1$, find $\dfrac{dy}{dx}$.

EXERCISE 11.1. Consider the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Find dy/dx using the technique of *implicit differentiation*.

EXERCISE 11.2. Consider the curve defined by the equation $x^2y^7 - x^3y^2 = 1$. Calculate dy/dx.

11.3. Higher-Order Derivatives

Now let's address the problem of calculating higher-order derivatives using implicit differentiation.

EXAMPLE 11.4. Consider $x^2 + y^2 = 1$. Calculate $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, and $\dfrac{d^3y}{dx^3}$ using implicit differentiation.

Summary of the Technique. Here is the synthesis of the method for calculating higher-order derivatives using implicit differentiation.

1. Given an equation: $F(x,y) = c$. Differentiate implicitly; you will obtain another equation $F_0(x,y,y') = 0$. Solve this equation for y' to obtain the solution:

$$y' = F_1(x,y). \tag{10}$$

   This means that y' is represented in terms of x and y.

2. Take equation (10) and differentiate it with respect to x. You will obtain an equation of the general form:

$$y'' = F_2(x,y,y'). \tag{11}$$

   Take your answer for y' in (10) and substitute it into (11) to obtain something like:

$$y'' = F_3(x,y), \tag{12}$$

   that is, work to write your answer in terms of the basic variables x and y.

3. To calculate the third derivative, y''', with respect to x, take (12) and differentiate both sides with respect to x to obtain an expression:
$$y''' = F_4(x, y, y').$$
As before, take (10) and substitute for y' into this equation. We will arrive at the equation:
$$y''' = F_5(x, y). \tag{13}$$
This represents y''' in terms of x and y.
4. For higher derivatives, you would continue in the same manner.

EXERCISE 11.3. In the spirit of the above outline, write out the steps for calculating y'''', also known as $y^{(4)}$.

EXERCISE 11.4. Let $\sin(y) = x$. Calculate $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, and $\dfrac{d^3y}{dx^3}$ using the method of implicit differentiation. In all cases, leave the answer in terms of x and y.

12. The Mean and the Extreme

This section still under construction. It is my intention to develop a series of topics covering theory and techniques of finding local and absolute extrema.

In this section we discuss the Extreme Value Theorem, which is a fundamental tool in optimization problems, and the Mean Value Theorem, which has many theoretical consequences and many useful applications.

12.1. Extrema: Introductory Concepts

You know, as someone who has seen many graphs, that some curves have *high points* and/or *low points*. In this section we give a formal definition these terms and present a famous theorem that speaks towards the existence of these high and low points.

Actually, there are two types of *extreme points* on a graph: *local extreme points* and *absolute extreme points*. The term 'extreme point' refers to high or low points on a graph. Let's have the formal definitions.

Definition 12.1. Let f be a function defined on an interval I, and let $c \in I$.

(1) We say that f has a *local or relative minimum* at $c \in I$ provided there is a open interval J containing c such that for all $x \in J \cap I$, $f(x) \geq f(c)$.
(2) We say that f has a *local or relative maximum* at $c \in I$ provided there is a open interval J containing c such that for all $x \in J \cap I$, $f(x) \leq f(c)$.

Definition Notes: The interval J in these definitions is used to describe a meaning of "closeness" to the number c. Basically, (1) in **Definition 12.1** states that c is a local minimum provided there is a closeness, J, to c, such that if x is that close ($x \in J$) and if x is within the domain of definition of f ($x \in I$), then $f(x) \leq f(c)$. That is, everywhere "close" to c, $f(c)$ is the smallest value of the function.

- The condition $x \in J \cap I$ simply requires x to be "close" to c $(x \in J)$ and x be in the domain of f $(x \in I)$. ■

EXERCISE 12.1. Why do you suppose in Definition 12.1 we required J to be an open interval?

Of considerable more applied interest is the notion of *absolute extrema*, the definitions of which follow.

Definition 12.2. Let f be a function having domain $\operatorname{Dom}(f)$, and let $c \in \operatorname{Dom}(f)$.

(1) We say that $c \in \operatorname{Dom}(f)$ is an *absolute minimum of f over* $\operatorname{Dom}(f)$ if for all $x \in \operatorname{Dom}(f)$, $f(x) \geq f(c)$.
(2) We say that $c \in \operatorname{Dom}(f)$ is an *absolute maximum of f over* $\operatorname{Dom}(f)$ if for all $x \in \operatorname{Dom}(f)$, $f(x) \leq f(c)$.

Definition Notes: In the definition, $\operatorname{Dom}(f)$ is not necessarily the natural domain of the function; it may be a subset of the natural domain of the function f. For example, the function might be $f(x) = x^2 - x$, whose natural domain of definition is $\mathbb{R}$, but may be interested in the absolute maximum (or minimum) of this function over the interval $\text{Dom}(f) = [\,0, 1\,]$.

■ It is important to remember that the notion of absolute maximum or minimum is itself relative concept — relative to the current domain of definition of f. As a trivial and simple example of this remark, consider the function $f(x) = x$ and $\text{Dom}(f) = [\,0, 1\,]$. In this case, the absolute minimum is at $c = 0$ and the actual minimum value is $f(0) = 0$. If we take the same function and change the domain to $\text{Dom}(f) = [\,1, 2\,]$, then the absolute minimum occurs at $c = 1$ and the minimum value is $f(1) = 1$. ■

Theorem 12.3. (Fermat's Theorem) *Let f be a function defined on an interval I. Suppose $c \in I$ is a local extremum such that c is not an endpoint of I, and $f'(c)$ exits, then $f'(c) = 0$.*

Proof.

Theorem Notes: The proof is easy to understand and it is strongly recommended that the student study it.

■ Now we turn to the larger question of locating local extrema. Let $c \in I$ be any local extrema of f, then either c is an endpoint, or not; further, in the case c is not an endpoint, either $f'(c)$ exists, or $f'(c)$ does not exist. In the former case, we know from FERMAT'S THEOREM, $f'(c) = 0$.

■ To summarize, then, if $c \in I$ is any local extrema of f, then either c is an endpoint, $f'(c) = 0$, or $f'(c)$ does not exist. Should you be looking for local extrema, these would be the three places to look.

■

Before we continue, let's define a useful terminology.

Definition 12.4. Let f be a function and let $c \in \text{Dom}(f)$, then c is called a *critical point* of f if either $f'(c)$ does not exist, or $f'(c) = 0$.

Definition Notes: A given critical point can be classified into any one of three categories: local maximum, local minimum, or neither. In the latter case, that is, if c is neither a local maximum nor minimum, very often, with the kind of functions we deal with, this means that c is a *saddle point*. Examples are listed below. ■

Here is a set of questions that occurs to me. Given that we have a critical point c in hand:

1. How can we tell, in a *sightless way* whether the critical point is a maximum, minimum, or neither.
2. How can we tell, in a *sightless way* whether the critical point is an absolute extrema or a local extrema?

Notice that an emphasis is placed on *sightless techniques*. In the age of the modern graphing calculator these questions can be answered by simply graphing the function. These kinds of techniques are certainly important at our level of play, but I am anticipating a more general setting. In many applied problems it is desired to find maximums and minimums of functions of many variables. In this case, there is no geometry — one cannot graph the function and visually see the maximums or minimums. We need to start developing techniques, therefore, that will generalized to multidimensions where there are not visualizations.

12.2. The Extreme Value Theorem

In this section we introduce the title theorem and give series of procedural steps for locating absolute extrema of a function over closed and bounded intervals.

We want to hunt for the absolute extrema of a function over a specific interval. It would be nice to know that the absolute extrema exist *before* we start to look for them — this would save a lot of time, pain, and mental anguish. The next theorem describes under what conditions we will *know* that absolute maximums and minimums exist.

Theorem 12.5. (The Extreme Value Theorem) *Let f be a continuous function defined over a closed interval $I = [\,a, b\,]$. Then*

(1) *there exists (at least one) point $x_{\min} \in I$ such that f has an absolute minimum at $x_{\min}$;*
(2) *there exists (at least one) point $x_{\max} \in I$ such that f has an absolute maximum at $x_{\max}$.*

Proof. Beyond the scope of these notes.

Theorem Notes: If $x_{\min}$ is a point at which f has an absolute minimum, then the minimum value is $f(x_{\min})$. In this case, sometimes we write,

$$\min_{a \le x \le b} f(x) = f(x_{\min});$$

the meaning of the notation is self-evident.

▪ Similarly, if $x_{\max}$ is a point at which f has an absolute maximum, then the maximum value is $f(x_{\max})$. In this case, we write,

$$\max_{a \le x \le b} f(x) = f(x_{\max});$$

▪ As you know from your own graphing experiences, an absolute maximum (minimum) may occur at several points (or at infinitely many points). Think of the function $f(x) = \cos(x)$, $I = [\,0, 4\pi\,]$. It should be clear to you that

$$\max_{0 \le x \le 4\pi} \cos(x) = 1,$$

and that the maximum value is *attained* at $x = 0$, 2π, and 4π. In the same way,

$$\min_{0 \leq x \leq 4\pi} \cos(x) = -1,$$

and that the minimum value is *attained* at $x = \pi$ and 3π. ■

The Extreme Value Theorem and Fermat's Theorem together enable us to setup a definite procedure for finding the absolute extrema of a continuous function f defined over a closed and bounded interval $I = [\,a, b\,]$.

Problem. Given a *continuous function* f defined on a closed and bounded interval $[\,a, b\,]$, we want to find the absolute maximum and absolute minimum of the function f *over* the interval $[\,a, b\,]$.

The Method. We proceed as follows:

1. Find the critical points of f over $[\,a, b\,]$: these are the numbers, x, at which $f'(x) = 0$ or $f'(x)$ does not exist. Let me represent these numbers symbolically:

$$x_1, x_2, x_3, \ldots, x_n.$$

2. Include the endpoints a and b is the above list:

$$a, x_1, x_2, x_3, \ldots, x_n, b. \tag{1}$$

3. Calculate the value of f at each of the numbers in the list (1).

x	$f(x)$
a	$f(a)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
$\vdots$	$\vdots$
x_n	$f(x_n)$
b	$f(b)$

4. The absolute maximum is the largest number in the right column and the absolute minimum is the smallest number in the right column of the above table.

Justification of this Method. Here is a detailed explanation of the reasoning behind this method.

EXAMPLE 12.1. Consider the function $f(x) = 3 + 4x - 3x^3$ restricted to the interval $[-1, 2]$. Find the absolute maximum and absolute minimum of f over the interval f and state at what values of x these extrema are attained.

12.3. The Mean Value Theorem

In the previous sections we introduced definitions of local and absolute extrema as well as some supporting theory that enable us to develop a method of locating absolute extrema. In this section, we continue to develop some underlying theory that will enable us to develop *analytic* and *sightless* methods of classifying critical points.

We begin by presenting a theorem that is later used to prove the MEAN VALUE THEORY, but is itself of considerable interest.

Theorem 12.6. (Rolle's Theorem) *Let f be a function be continuous on the interval $[a, b]$ and differentiable on the interval (a, b) such that*

$$f(a) = f(b)$$

Then there exists a number $c \in (a, b)$ *such that* $f'(c) = 0$.

Proof.

Let f be a differentiable function. Then

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

This equation establishes a relationship between the values of the function f near x and the value of the derivative f' at that same x. The relationship is a complex one though: The two are connected by the limit process. In this section we explore the Mean Value Theorem, a theorem which establishes another relationship between f and f'.

The Mean Value Theorem is an “obvious” truth, but like many truths in mathematics, it is more difficult to prove than you would suppose. Let me illustrate this obvious fact through an example.

Example 12.2. This example illustrates the Mean Value Theorem using a skier and a ski slope.

Now let me formally state the MEAN VALUE THEOREM. See if you can recognize the skier analogy in the theorem.

Theorem 12.7. (The Mean Value Theorem) *Let f be a function be continuous on the interval $[\,a, b\,]$ and differentiable on the open interval $(\,a, b\,)$. Then there exists a number $c \in (\,a, b\,)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{3}$$

or,

$$f(b) - f(a) = f'(c)(b - a) \tag{4}$$

or even,

$$f(b) = f(a) + f'(c)(b - a) \tag{5}$$

Proof.

Theorem Notes: It is equation (3) that yields the skiing interpretation of EXAMPLE 12.2, do you see it?

▪ The point $T(\,a, f(a)\,)$ is the left-most point on the graph of f, the point $B(\,b, f(b)\,)$ is the right-most point on the graph of f. What is the interpretation of right-hand side of (3)? It is the slope of the line that passes through T and B — the gradient of the curve. According to the MEAN VALUE THEOREM, there is some $c \in (a, b)$ such that (3) is true. Recall that $f'(c)$ is the slope of the line tangent to the graph at the point $x = c$. Thus, (3) states that there is some point on the graph ($x = c$) at which the tangent is parallel to the gradient of the graph. This is the content of EXAMPLE 12.2.

■ The equation (4) is an algebraic variation of (3). It basically gives a direct relationship between the values of the function f, and the values of the derivative of the function f'. In subsequent studies, we will take advantage of (4).

■ When the MEAN VALUE THEOREM is used in the form of (5) it usually means that we are interested in relating the value $f(b)$ to $f(a)$.

■ The phrasing of THEOREM 12.7 is a little strange, "Suppose f is ... continuous on $[a, b]$ and differentiable on (a, b)." why do you suppose that is? ■

The next theorem is a simple, yet important, application of the MEAN VALUE THEOREM.

Theorem 12.8. *Suppose f is a function that is continuous on the interval $[\,a,b\,]$ and differentiable on the open interval $(\,a,b\,)$ such that*

$$f'(x) = 0 \text{ for all } x \in (\,a,b\,),$$

then f is constant over the interval $[\,a,b\,]$; i.e., there is some constant, C, such that

$$f(x) = C \text{ for all } x \in [\,a,b\,].$$

Proof. The proof is simple enough to be presented "in-line."

Suppose f is a function such that $f'(x) = 0$ for all $x \in (\,a,b\,)$. Define $\mathrm{C} := f(a)$. Claim $f(x) = \mathrm{C}$ for all $x \in [\,a,b\,]$.

To that stated end, choose any $x \in (\,a,b\,]$ and apply the MEAN VALUE THEOREM to our function f but over the interval $[\,a,x\,]$. By the MEAN VALUE THEOREM, there is some $c \in (\,a,x\,)$ such that

$$f(x) - f(a) = f'(c)(x-a). \tag{6}$$

But we are assuming that f' is identically zero; therefore, $f'(c) = 0$. Updating (6) using this information we get

$$f(x) - f(a) = (0)(x - a) = 0$$

or,

$$f(x) = f(a)$$

But, $f(a) = \mathrm{C}$, so

$$f(x) = C$$

which is what we wanted to prove. Hurray! $\square$

Corollary 12.9. *Suppose f and g are continuous over the interval $[\,a, b\,]$ and differentiable over the open interval $(\,a, b\,)$ such that*

$$f'(x) = g'(x) \text{ for all } x \in (\,a, b\,),$$

then there is a constant, C, such that

$$f(x) = g(x) + C \text{ for all } x \in (\,a, b,\,).$$

Proof. Define a new function by

$$H(x) = f(x) - g(x) \qquad x \in [\,a, b\,].$$

Then,

$$H'(x) = f'(x) - g'(x) = 0 \qquad x \in [\,a, b\,].$$

Now, by Theorem 12.8, there is a constant, C, such that $H(x) = \mathrm{C}$. This means

$$\mathrm{C} = H(x) = f(x) - g(x) \qquad x \in [\,a, b\,]$$

or,

$$f(x) = g(x) + \mathrm{C} \qquad x \in [\,a, b\,]$$

which is the advertised equation. □

Theorem Notes: Corollary 12.9 states that the only way two functions can have exactly the same derivative is for the graph of one of the functions to be a vertical translation of the graph of the other.

▪ This corollary is a foundation stone of Indefinite Integration. Watch of reference to this corollary when you study integration. ▪

12.4. Criteria for Monotonicity

Let's begin with a phalanx of definitions.

Definition 12.10. Let f be defined on an interval I.

(1) The function f is *strictly increasing* over the interval I provided

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \implies f(x_1) < f(x_2).$$

(2) The function f is *strictly decreasing* over the interval I provided

$$x_1, x_2 \in I \text{ and } x_1 < x_2 \implies f(x_1) > f(x_2).$$

Definition Notes: A *monotone function* is one that is strictly increasing or decreasing. Thus, if I say that "f is monotone over the interval (a, b), I mean that f is either strictly increasing or strictly decreasing over that interval.

■ A function may be strictly increasing (decreasing) over its entire domain, or a function may decrease over portions of the domain and increase over other portions of the domain. ■

EXAMPLE 12.3. The function $f(x) = x$ is strictly increasing over its entire natural domain, $\mathbb{R}$.

EXAMPLE 12.4. The function $f(x) = x^2$ is strictly increasing over the interval $(0, +\infty)$ and is strictly decreasing over the interval $(-\infty, 0)$.

As was seen by these two illustrative examples, the increasing or decreasing nature of a function can be proven using algebra—but algebraic methods quickly become too difficult to use even for "moderately complicated" functions.

Typically, there are two approaches of verifying that a function increases or decreases over a specific interval:

1. *Visual or graphical Methods.* (If we graph the function we can "see" where the function increases or decreases.)
2. *Analytical Methods.*

a. *Algebraic Methods* (as illustrated in the last two examples).
b. *Calculus Methods*.

In this section, of course, we are primarily interested in analytic methods using *Calculus*!

The next theorem states the relevant theory that leads to our analytic methods.

Theorem 12.11. (The Monotonicity Theorem) *Let f be continuous on $[\,a, b\,]$ and differentiable over the open interval $(\,a, b\,)$.*

(1) Test for Increasing. *If $f'(x) > 0$ over the interval $(\,a, b\,)$, then f is strictly increasing over the interval $[\,a, b\,]$.*
(2) Test for Decreasing. *If $f'(x) < 0$ over the interval $(\,a, b\,)$, then f is strictly decreasing over the interval $[\,a, b\,]$.*

Proof. The proof is a simple consequence of the much celebrated MEAN VALUE THEOREM and can be easily understood by the reader. You are invited to read it.

Theorem Notes: Theorem 12.11 essentially gives a way of detecting the monotone nature of a differentiable function through the examination of its first derivative. We simply attempt to find everywhere the first derivative of the function is positive—that will generally over an interval of numbers—and by (a) above, the parent function f will be increasing over that same interval. ■

Let's begin with some simple examples ... then it's your turn.

EXAMPLE 12.5. Find all intervals of increase and decrease of the function $f(x) = x^4 - 8x^2 + 2$.

This example illustrates the way in which Theorem 12.11 is applied. To find the intervals of increase and decrease of a function f, calculate f' and do a sign analysis on the function f' — typically this is done using the *Sign Chart Method*.

Example 12.6. Find the intervals of increase and decrease of the function $f(x) = x^3(x-2)^4$.

12.5. Classifying Critical Points: The First Derivative Test

The first derivative, in addition to containing monotone information of a function, also contains information leading us to classify critical points. In fact it is the monotone information that makes the classification.

Theorem 12.12. (The First Derivative Test) *Let f be a function and c a critical point of f.*

(1) Test for Local Maximum. *Suppose f' is positive to the left of c and negative to the right of c, then c is a local maximum.*
(2) Test for Local Minimum. *Suppose f' is negative to the left of c and positive to the right of c, then c is a local minimum.*
(3) Test for a Saddle Point. *Suppose f' does not change signs at c, then c is a saddle point.*

Proof.

Let's analyze the function in Example 12.5 and classify all critical points.

Example 12.7. Find and classify all critical points of the function $f(x) = x^4 - 8x^2 + 2$.

The function we examined in Example 12.6 is interesting as it provides an example of a saddle point. The next example examines this function once again.

Example 12.8. Find and classify all critical points of the function $f(x) = x^3(x-2)^4$.

Solutions to Exercises

8.1. Step 1: recognize and realize that the given function is a composition of two other functions. Step 2: decompose. Step 3: apply chain rule.

The function $F(x) = (3x^4 + 5x)^{1/2}$ is the composition of two functions: the *outer function*: $f(x) = x^{1/2}$; and the *inner function*: $g(x) = 3x^4 + 5x$.

Preliminary Calculations:

$$
\begin{aligned}
f(x) &= x^{1/2} & \qquad g(x) &= 3x^4 + 5x \\
f'(x) &= \frac{1}{2}x^{-1/2} & \qquad g'(x) &= 12x^3 + 5.
\end{aligned}
$$

Therefore, by the Chain Rule we obtain,

$$\begin{aligned} F'(x) &= (f \circ g)'(x) = f'(g(x))g'(x) \\ &= f'(3x^4 + 5x)(12x^3 + 5) \\ &= \frac{1}{2}(3x^4 + 5x)^{-1/2}(12x^3 + 5). \end{aligned}$$

Thus,

$$\boxed{F'(x) = \frac{1}{2}(12x^3 + 5)(3x^4 + 5x)^{-1/2}.}$$

Did you get it?

Exercise 8.1. ■

8.2. The function F is obviously the composition of two functions:

$$y = f(x) \qquad g(x) = 1/x^4 = x^{-4}.$$

Make the standard calculations.

$$\begin{aligned} f(x) &\qquad g(x) = x^{-4} \\ f'(x) &\qquad g'(x) = -4x^{-5}. \end{aligned}$$

Therefore, by the Chain Rule we obtain,

$$\begin{aligned} F'(x) = (f \circ g)'(x) &= f'(g(x))g'(x) \\ &= f'(1/x^4)(-4x^{-5}) \\ &= -4\frac{f'(1/x^3)}{x^5} \end{aligned}$$

Thus,

$$\boxed{F'(x) = -4\frac{f'(1/x^3)}{x^5}.}$$

Exercise 8.2. ■

8.3. The function F is obviously the composition of two functions: one abstract and the other specific. Define $f(x) = x^6$, then

$$F(x) = [g(x)]^6 = f(g(x)).$$

Of course, $f'(x) = 6x^5$ and so, by the Chain Rule, we have

$$F'(x) = f'(g(x))g'(x) = 6[g(x)]^5 g'(x)$$

Thus,

$$\boxed{F'(x) = 6[g(x)]^5 g'(x).}$$

Exercise Notes: This formula gives us a way of differentiating *any* base function g raised to the sixth power. A useful formula indeed, if you have the sixth power!

- The formula is only valid when we have a base function raised to the sixth power. What if we wanted to differentiate a function of the form $F(x) = [g(x)]^7$?
- In the section The Power Rule Revisited, a general formula is obtained for differentiating any function of the form $F(x) = [g(x)]^r$, where r is any exponent.

■ Maybe you can develop the formula yourself *before* getting this section. (*Hint*: Use this exercise as a template; change the exponent of 6 to an abstract exponent of r and follow the development line-by-line, making all appropriate changes.) ■

Exercise 8.3. ■

8.4. The function is $y = (3x^3 - 6x)^{1/2}$. Decompose using a substitution technique. Let $u = 3x^3 - 6x$, then $y = u^{1/2}$. Thus

$$y = u^{1/2} \qquad u = 3x^3 - 6x$$
$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \qquad \frac{du}{dx} = 9x^2 - 6.$$

By the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{2}u^{-1/2}(9x^2 - 6). \end{aligned} \tag{A-1}$$

Again, typical of this technique, the answer comes out in terms of x *and* u. A re-substitution will solve this problem: $u = 3x^3 - 6x$. Substituting this into (A-1) we obtain

$$\boxed{\frac{dy}{dx} = \frac{1}{2}(3x^3 - 6x)^{-1/2}(9x^2 - 6)}$$

The answer could be cleaned up a little.

$$\boxed{\frac{dy}{dx} = \frac{3(3x^2 - 2)}{2\sqrt{3x^3 - 6x}}}$$

Exercise 8.4. ■

8.5. We want to differentiate a function raised to a power.

$$\frac{d}{dx}(4x^2 + 1)^{23} = 23(4x^2 + 1)^{22}(8x) = 184x(4x^2 + 1)^{22}.$$

Exercise 8.5. ■

8.6. We want to differentiate a function raised to a power.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x}{1+x}\right)^5 &= 5\left(\frac{x}{1+x}\right)^4 \frac{d}{dx}\frac{x}{1+x} && \text{Power} \\
&= 5\left(\frac{x}{1+x}\right)^4 \frac{(1+x)(1)-x(1)}{(1+x)^2} && \text{Quot.} \\
&= 5\left(\frac{x}{1+x}\right)^4 \frac{1}{(1+x)^2} \\
&= \frac{5x^4}{(1+x)^6}
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}\left(\frac{x}{1+x}\right)^5 = \frac{5x^4}{(1+x)^6}.}$$

Exercise 8.6. ■

8.7. This is the derivative of a product of two functions. Use the Product Rule.

$$\begin{aligned}\frac{d}{dx}x^4(2x+1)^{3/2} &= x^4\frac{d}{dx}(2x+1)^{3/2} + 4x^3(2x+1)^{3/2}\\ &= x^4(3/2)(2x+1)^{1/2}(2) + 4x^3(2x+1)^{3/2}\\ &= x^3(2x+1)^{1/2}(3x+4(2x+1))\\ &= x^3(2x+1)^{1/2}(11x+4)\end{aligned}$$

The ultimate in answers is

$$\boxed{\frac{d}{dx}x^4(2x+1)^{3/2} = x^3(11x+4)(2x+1)^{1/2}.}$$

Notes: If you have been seriously reading and working along with these notes, you will, hopefully, begin to see a *style* of solution to these differentiation problems. All of them are pretty much the same. ■

The secret? Use good notation, classify the function types (product, quotient, sum, etc.), know the rules (the formulas), use good algebra.

That's all.

Exercise 8.7. ■

8.8. We are asked to differentiate a quotient of two functions of s. You should have used the Quotient Rule.

Using the Quotient Rule and the Power Rule all in one magnificent step we get,

$$\frac{d}{ds}\frac{(s+1)^3}{(2s+1)^5} = \frac{(2s+1)^5(3)(s+1)^2 - (s+1)^3(5)(2s+1)^4(2)}{(2s+1)^{10}} \quad \text{(A-2)}$$

This is the initial macro expansion of the quotient and product rules. This answer represents a *minimal* response to the question. However, you should have the drive, the power, the ability to improve on this answer. I hope this answer is unacceptable to you.

Assignment: Invoke your *algebra module of knowledge* and simplify. Below is a multiple choice quiz, after you have finished simplifying, make your choice. Do *not* look at the choices before your have finished your simplifications. *Simplify Now*!

After you simplify, go to the next page for a little quiz!

Which of the following is a simplification of (A-2)?

(a) $\dfrac{(s+1)^2}{(2s+1)^{10}}$

(b) $\dfrac{3(s+1)^3(4s+7)}{(2s+1)^{10}}$

(c) $-\dfrac{(4s+7)(s+1)^2}{(2s+1)^6}$

(d) $-\dfrac{(4s+7)(s+1)^2}{(2s+1)^{10}}$

Exercise 8.8. ■

9.1. Let's have a quiz.

Which of the following is the $\dfrac{d}{dx} 3\cos(x)$?

(a) $3\cos(x)$ (b) $-3\sin(x)$ (c) $3\sin(x)$ (d) $-3\cos(x)$

Which of the following is the $\dfrac{d}{dt} - 5\sec(t)$?

(a) $-5\sec^2(x)$ (b) $-5\csc(x)\cot(x)$
(c) $5\cot^2(x)$ (d) $-5\sec(x)\tan(x)$

Hopefully, you were two for two. Exercise 9.1. ■

9.2. Proceed as follows,

$$\begin{aligned} \frac{d}{dx}\tan(x)\Big|_{x=\pi/3} &= \sec^2(x)\big|_{x=\pi/3} \qquad \text{from (3)} \\ &= \sec^2(\pi/3) \\ &= 4 \end{aligned}$$

Recall

$$\sec^2(\pi/3) = \frac{1}{\cos^2(\pi/3)} = \frac{1}{(1/2)^2} = \frac{1}{1/4} = 4.$$

Now let's turn to the problem of calculating the equation of the line tangent to the graph at $x = \pi/3$.

In order to find any equation of a line we need two pieces of information: A point the line passes through; the slope of the target line.

Work through the rest of the problem through a series of steps.

The Point: Which of the following can be taken to be our point?

(a) $(\pi/3, \sqrt{3/2}\,)$ (b) $(\,\pi/3, 2\,)$ (c) $(\,\pi/3, 4\,)$ (d) $(\,\pi/3, \sqrt{3}\,)$

The Slope: Which of the following is the slope of the line tangent to the graph of $y = \tan(x)$.

(a) $1/2$ (b) 2 (c) $\sec^2(x)$ (d) 4

The Equation: Which of the following is the equation of the line tangent to the graph at $x = \pi/3$?

(a) $y - \dfrac{\sqrt{3}}{2} = \sqrt{3}(x - \frac{\pi}{3})$ (b) $y - \sqrt{3} = 4(x - \frac{\pi}{3})$

(c) $y - \sqrt{3} = \sec^2(x)(x - \frac{\pi}{3})$ (d) $y - \frac{\pi}{3} = 4(x - \sqrt{3})$

Exercise 9.2. ■

9.3. We first recall that

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Therefore, we use the Quotient Rule.

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\
&= \frac{\cos(x)\frac{d}{dx}\sin(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

As you can see this is a straight forward enough calculation. So it goes with all the trig functions $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$. The difficult task was to calculate the derivatives of $\sin(x)$ and $\cos(x)$, the other four trig functions are ratios of these two.

Exercise 9.3. ■

9.4. The function is $F(x) = \sin(6x^3)$. You should have realized that F is the composition of two more elementary functions: the outer function is $f(x) = \sin(x)$ and the inner function is $g(x) = 6x^3$.

Preliminary Calculations:

$$\begin{aligned} f(x) &= \sin(x) & \qquad g(x) &= 6x^3 \\ f'(x) &= \cos(x) & \qquad g'(x) &= 18x^2. \end{aligned}$$

Therefore, by the Chain Rule we obtain,

$$\begin{aligned} F'(x) = (f \circ g)'(x) &= f'(g(x))g'(x) \\ &= f'(6x^3)(18x^2) \\ &= \cos(6x^3)(18x^2). \end{aligned}$$

Thus,

$$\boxed{F'(x) = 18x^2 \cos(6x^3).}$$

Exercise 9.4. ■

9.5. The problem $\dfrac{d}{dx}\tan(\sin(x))$ is nothing more than the derivative of the tangent function of some function of x. Use Trig formula (3),

$$\begin{aligned}\frac{d}{dx}\tan(\sin(x)) &= \sec^2(\sin(x))\frac{d}{dx}\sin(x)\\ &= \sec^2(\sin(x))\cos(x)\end{aligned}$$

You simply have to learn to look at a function and classify it. In this case, we wanted to differentiate the *tangent of some function of* x. Keep it simple. Do the obvious.

The derivative of the tangent of some function of x is the secant squared of that same function of x, times the derivative of that function of x. This would be the verbalization of the tangent formula.

Exercise 9.5. ■

9.6. Problem: $\dfrac{d}{dx}\sin(\tan^4(3x^3))$. Initially, we want to differentiate the *sine of some function of* x — that's all. Apply (1),

$$\frac{d}{dx}\sin(\tan^4(3x^3)) = \cos(\tan^4(3x^3))\frac{d}{dx}\tan^4(3x^3)$$

Now to continue, we must analyze our next differentiation problem. Are we to differentiate the tangent of some function of x, or are we to differentiate a function raised to a power? The notation $\tan^4(3x^3)$, is short-hand for $(\tan(3x^3))^4$. This is a composition of two functions with the 4^{th}-power function the "outer" function; consequently, we see the expression $\tan^4(3x^3)$ as a *function raised to a power* — hence, apply the Power Rule. Continuing the calculation now,

$$\begin{aligned}\frac{d}{dx}\sin(\tan^4(3x^3)) &= \cos(\tan^4(3x^3))\frac{d}{dx}\tan^4(3x^3)\\ &= \cos(\tan^4(3x^3))(4)\tan^3(3x^3)\frac{d}{dx}\tan(3x^3)\end{aligned}$$

We have another differentiation problem. This one is the derivative of the tangent of some function. Use (3) of the trig function set.

$$\begin{aligned}\frac{d}{dx}\cos(\tan^4(3x^3)) &= \cos(\tan^4(3x^3))(4)\tan^3(3x^3)\frac{d}{dx}\tan(3x^3)\\ &= \cos(\tan^4(3x^3))(4)\tan^3(3x^3)\sec^2(3x^3)\frac{d}{dx}(3x^3)\\ &= \cos(\tan^4(3x^3))(4)\tan^3(3x^3)\sec^2(3x^3)(9x^2)\end{aligned}$$

The final answer made to look cute is

$$\boxed{\frac{d}{dx}\sec(\tan^4(3x^3)) = 36x^2\sec^2(3x^3)\tan(3x^3)\cos(\tan^4(3x^3)).}$$

That's quite a mouthful!

Exercise Notes: Actually, these differentiation problems are not that hard, IF you follow the rules exactly as they are laid out. Analyze the function type, apply the rules, slowly, methodically, one step at a time just as I have done. You'll come through o.k.

■ Perhaps now you can see from where the term "Chain Rule" comes. When you differentiate a function that is the composition of a large number of other functions, as was the case in this exercise, we differentiate the "outer" function times the derivative of the "inner" function. As a result, we generate a series of derivatives that are chained together.

■ Sometimes I say that the process of differentiation is the process of moving the d/dx symbol from left to right. Can you see why? ■

Exercise 9.6. ■

10.1. The first thing to know, is how to read the formula

$$y^{(n)} = 2^n \sin(2x + \frac{n\pi}{2}) \qquad n = 1, 2, 3, 4, \dots. \tag{A-3}$$

Write (A-3) for the cases of $n = 1, 2, 3, 4,$ *dotsc*.

$$\begin{aligned} n &= 1 & y' &= 2\sin(2x + \frac{\pi}{2}) \\ n &= 2 & y'' &= 4\sin(2x + \pi) \\ n &= 3 & y''' &= 8\sin(2x + \frac{3\pi}{2}) \\ n &= 4 & y^{(4)} &= 16\sin(2x + 2\pi) \end{aligned} \tag{A-4}$$

Compare these first derivatives with the results of (1). Except for the powers of 2, they don't compare well do they. However, if you apply the *additive rule* for the sine function

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(A)$$

to the equations in (A-4), I hope you'll change your mind.

Exercise 10.1. ■

10.2. The function $y\dfrac{x}{2x+1}$ is simply the quotient of two polynomials; it should be the height of triviality to differentiate.

The first derivative:

$$\begin{aligned} y' &= \frac{d}{dx}\frac{x}{2x+1} \\ &= \frac{(2x+1)(1) - x(2)}{(2x+1)^2} && \text{Quot. Rule} \\ &= \frac{1}{(2x+1)^2} \\ &= (2x+1)^{-2} \end{aligned}$$

The second derivative:

$$\begin{aligned} y'' &= \frac{d}{dx}(2x+1)^{-2} \\ &= (-2)(2x+1)^{-3}(2) && \text{Power Rule} \\ &= -4(2x+1)^{-3} \end{aligned}$$

The third derivative:

$$\begin{aligned} y''' &= -4\frac{d}{dx}(2x+1)^{-3} \\ &= (-4)(-3)(2x+1)^{-4}(2) \qquad \text{Power Rule} \\ &= 24(2x+1)^{-4} \end{aligned}$$

The ambitious student might calculate another half-dozen derivatives for practice. Exercise 10.2. ■

11.1. We follow standard procedures.

Treat y as it were a function of x, and differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = \frac{d\,1}{dx}.$$

Apply the differentiation rules, treating y as a function of x:

$$\frac{1}{4}(2x) + \frac{1}{9}(2y)\frac{dy}{dx} = 0.$$

Solve the equation for dy/dx:

$$\begin{aligned} \frac{2}{9}y\frac{dy}{dx} &= -\frac{1}{2}x \\ \frac{dy}{dx} &= -\frac{9x}{4y}. \end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = -\frac{9x}{4y}.}$$

You'll notice that dy/dx becomes infinite when $y = 0$. This usually indicates vertical tangent lines. The points on the ellipse corresponding to $y = 0$ are $(2, 0)$ and $(-2, 0)$. These two points are the vertices on the x-axis and we see that at these two points, we indeed have vertical tangents.

Maple Plot. A plot of this equation can be obtained from Maple using the commands:

```
> with(plots);

> implicitplot( x^2/4 + y^2/9 = 1, x=-2..2, y=-3..3);
```

Exercise 11.1. ■

11.2. The given equation $x^2y^7 - x^3y^2 = 1$ is an equation involving the variables x and y. We cannot explicitly solve for y in terms of x; therefore, we use the method of *implicit differentiation*.

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2y^7 - x^3y^2) = \frac{d\,1}{dx}.$$

Apply the differentiation rules wherever applicable, Step 2, all the while treating y as if it were a function of x, Step 1:

$$\frac{d}{dx}(x^2y^7) - \frac{d}{dx}(x^3y^2) = 0$$

$$(x^2\frac{d\,y^7}{dx} + y^7\frac{d\,x^2}{dx}) - (x^3\frac{d\,y^2}{dx} + y^2\frac{d\,x^3}{dx}) = 0$$

$$(x^2(7y^6\frac{dy}{dx}) + y^7(2x)) - (x^3(2y\frac{dy}{dx}) + y^2(3x^2) = 0$$

$$7x^2y^6\frac{dy}{dx} + 2xy^7 - 2x^3y\frac{dy}{dx} - 3x^2y^2 = 0.$$

Now, solve algebraically for $\frac{dy}{dx}$, Step 3:

$$7x^2y^6\frac{dy}{dx} + 2xy^7 - 2x^3y\frac{dy}{dx} - 3x^2y^2 = 0$$

$$(7x^2y^6 - 2x^3y)\frac{dy}{dx} = 3x^2y^2 - 2xy^7$$

$$x^2y(7y^5 - 2x)\frac{dy}{dx} = xy^2(3x - 2y^5)$$

Thus,

$$\frac{dy}{dx} = \frac{x^2y(7y^5 - 2x)}{xy^2(3x - 2y^5)},$$

or,

$$\boxed{\frac{dy}{dx} = \frac{x(7y^5 - 2x)}{y(3x - 2y^5)}.}$$

This then is the derivative of y with respect to x.

Question: Can you find a point, (x, y) that satisfies the equation $x^2y^7 - x^3y^2 = 1$? What the graph look like? This is a job for Maple! These questions are answered in the example contained in our discussion of Differentiation using Maple. Exercise 11.2. ■

11.3. In the spirit of the outline, you should have said to take equation (13) and differentiate it with respect to x, the result of which would be

$$y^{(4)} = F_6(x, y, y').$$

Now, take (10), and substitute for y' in the above equation to obtain the desired derivative in its most fundamental form:

$$y^{(5)} = F_7(x, y).$$

Exercise 11.3. ■

11.4. You should have proceeded according to the outline above. Indeed,

First Derivative:

1. $\dfrac{d}{dx}\sin(y) = \dfrac{dx}{dx}$
2. $\cos(y)\dfrac{dy}{dx} = 1$
3. $\boxed{y' = \dfrac{dy}{dx} = \sec(y)}$

Second Derivative:

1. $y'' = \dfrac{dy'}{dx} = \dfrac{d}{dx}\sec(y)$
2. $y'' = \sec(y)\tan(y)\dfrac{dy}{dx}$
3. $y'' = \sec(y)\tan(y)\sec(y)$
4. $\boxed{y'' = \sec^2(y)\tan(y)}$

The third derivative is done similarly. I calculated it to be

$$y''' = \sec^2(y)(3\sec^2(y) - 2),$$

expressed entirely in terms of the secant function. Verify please!

Exercise 11.4. ■

12.1. Assume the same notation as in Definition 12.1 and we assume for the sake of discussion that c is a local minimum.

If J is required to be an open interval and $c \in J$, then c is not one of the endpoints, since the endpoints do not belong to the interval.

Why is this important in the description of the concepts of local maximum or minimum? To properly answer that question, let's consider two cases: c is an endpoint of the interval I and c is not an endpoint of I.

c is not an endpoint of I: In this case, when we compute $J \cap I$ we will get an interval whose left-hand endpoint is strictly less than c and whose right-hand endpoint is strictly greater than c. (Do you see why?) That is, $J \cap I$ will include numbers on *both sides* of c. This is the important point. The condition

$$f(x) \geq f(c) \qquad x \in J \cap I,$$

states that the graph of f is higher (than $f(c)$) to the left of c and higher to the right of c; this is descriptive of the concept of local minimum.

If J was not required to be open, then J could contain one of its endpoints. In this case the requirement that $c \in J$ could mean that c is just an endpoint of J; therefore, c would be an endpoint of the interval $J \cap I$. The condition

$$f(x) \geq f(c) \qquad x \in J \cap I,$$

may be only saying that f is higher on one side of a. Think about it. This would be counter to the concept of a low point on the graph. In fact, if we don't require J to be an open interval, then most any point on the graph is both a maximum and minimum! Can you see why? (Think of the function $f(x) = x^2$ and take $c = 1$, try to argue that $c = 1$ is both a local maximum and a local minimum by dropping the word "open" from Definition 12.1.)

c is an endpoint of I: Assume for the purpose of argument that c is the left-hand endpoint of the interval I. In this case, $J \cap I$ will be an interval whose left-hand endpoint is c as well. The condition

$$f(x) \geq f(c) \qquad x \in J \cap I,$$

says that f is higher to the right of c and says nothing of what goes on to the left of c. But if c is the left-hand endpoint of the domain of definition I, we don't care what goes on to the left of c because that is outside the domain of the function f.

In this case, it really doesn't matter whether J is open or not.

Exercise 12.1. ■

Solutions to Examples

8.1. This is a continuation of the *Theorem Notes*. It was seen there that $F = f \circ g$, where $f(x) = x^{10}$ and $g(x) = x^2 + 1$. Note that we have not relabeled the variables to suggest composition — sorry! The Chain Rule formula states that at any x

$$F'(x) = (f \circ g)'(x) = f'(g(x))g'(x).$$

Let's make the necessary calculations:

$$\begin{aligned} f(x) &= x^{10} & g(x) &= x^2 + 1 \\ f'(x) &= 10x^9 & g'(x) &= 2x \end{aligned}$$

therefore,

$$f'(g(x)) = f'(x^2 + 1) = 10(x^2 + 1)^9$$

Finally,

$$\begin{aligned} F'(x) = (f \circ g)'(x) &= f'(g(x))g'(x) \\ &= 10(x^2 + 1)^9\, 2x = 20x(x^2 + 1)^9. \end{aligned}$$

Thus,

$$\boxed{\frac{d(x^2+1)^{10}}{dx} = 10x(x^2+1)^9.}$$

Example Notes: That was relatively painless — I had no problems at all with it. The key point is the ability of the student to decompose functions (uncompose?), then differentiate each separately, and multiply them together. ■

Example 8.1. ■

8.2. Step 1: recognize and realize that the given function is a composition of two other functions. Step 2: decompose. Step 3: apply chain rule.

The function $F(x) = (3x^3 - x)^7$ is the composition of two functions: the *outer function*: $f(x) = x^7$; and the *inner function*: $g(x) = 3x^3 - x$.

Preliminary Calculations:

$$\begin{aligned} f(x) &= x^7 & \qquad g(x) &= 3x^3 - x \\ f'(x) &= 7x^6 & \qquad g'(x) &= 9x^2 - 1. \end{aligned}$$

Therefore,

$$\begin{aligned} F'(x) &= (f \circ g)'(x) = f'(g(x))g'(x) \\ &= f'(3x^3 - x)(9x^2 - 1) \\ &= 7(3x^3 - x)^6(9x^2 - 1). \end{aligned}$$

Thus,

$$\boxed{F'(x) = 7(9x^2 - 1)(3x^3 - x)^6.}$$

But wait, we wanted $F'(2)$ — mere calculation.

$$\boxed{F'(2) = 7(35)(22)^6.}$$

I leave the punching of calculator buttons to you, but the derivative is large.

Example 8.2. ■

8.3. In some sense, this is an easier problem than working with a specific function. The function F is obviously the composition of two functions:

$$y = f(x) \qquad g(x) = x^3.$$

It should be clear to you that $F = f \circ g$.

Now, proceeding along standard lines of inquiry we make some preliminary calculations:

$$\begin{array}{ll} f(x) & g(x) = x^3 \\ f'(x) & g'(x) = 3x^2. \end{array}$$

Notice that because the function f is an abstract differentiable function, it is the height triviality to differentiate it: the derivative of f is f'. Isn't that easy?

Therefore, by the Chain Rule we obtain,

$$\begin{aligned} F'(x) &= (f \circ g)'(x) = f'(g(x))g'(x) \\ &= f'(x^3)(3x^2) \\ &= 3x^2 f'(x^3). \end{aligned}$$

Thus,

$$\boxed{F'(x) = 3x^2 f'(x^3).}$$

Example 8.3. ■

8.4. The function is $y = (5x^4 - 12x^2)^3$. Decompose using a substitution technique. Let $u = 5x^4 - 12x^2$, then $y = u^3$. Thus

$$y = u^3 \qquad\qquad u = 5x^4 - 12x^2$$
$$\frac{d}{du}y = 3u^2 \qquad \frac{d}{dx}u = 20x^3 - 24x.$$

By the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (3u^2)(20x^3 - 24x)\\ &= 6xu^2(5x^2 - 6).\end{aligned} \tag{S-1}$$

The trouble with (S-1) is that the answer is in terms of x *and* u. We want the derivative of y with respect to x; usually, we want the answer *entirely* in terms of the independent variable, and that's x. It is simple enough to convert the answer in (S-1) to x — *just replace* u with what it is equal to in terms of x: $u = 5x^4 - 12x^2$.

Substitute $u = 5x^4 - 12x^2$ into (S-1) to obtain

$$\frac{dy}{dx} = 6x(5x^4 - 12x^2)^2(5x^2 - 6).$$

Thus, since $y = (5x^4 - 12x^2)^3$, we have shown that

$$\boxed{\frac{d}{dx}(5x^4 - 12x^2)^3 = 6x(5x^4 - 12x^2)^2(5x^2 - 6).}$$

Example Notes: Actually, the derivative can be simplified a bit more. If we wanted to be true to our algebraic roots, we would continue.

$$\boxed{\frac{d}{dx}(5x^4 - 12x^2)^3 = 6x^5(5x^2 - 6)(5x^2 - 12)^2.}$$

Verify please! ■

Example 8.4. ■

8.5. The problem is to calculate $\dfrac{d}{dx}(1-3x^3)^{10}$. The first observation you should make is that we are required to differentiate a function raised to a power. This is the first and most critical observation.

The second mental connection you need to make is this: Since we want to differentiate a function raised to a power, the Power Rule needs to be used first.

$$\begin{aligned}
\frac{d}{dx}(1-3x^3)^{10} &= 10(1-3x^3)^9\frac{d}{dx}(1-3x^3) && \text{Power Rule}\\
&= 10(1-3x^3)^9(-9x^2) && \text{ditto}\\
&= -90x^2(1-3x^3)^9
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}(1-3x^3)^{10} = -90x^2(1-3x^3)^9}$$

Example 8.5. ■

8.6. The first thing we need to do is to rewrite the function to be differentiated:

$$\frac{1}{(s^4 - s + 1)^{3/4}} = (s^4 - s + 1)^{-3/4}.$$

Now we can apply the power rule.

$$\begin{aligned} \frac{d}{ds}(s^4 - s + 1)^{-3/4} &= -\frac{3}{4}(s^4 - s + 1)^{-7/4}\frac{d}{ds}(s^4 - s + 1) \\ &= -\frac{3}{4}(s^4 - s + 1)^{-7/4}(4s^3 - 1) \end{aligned} \tag{S-2}$$

Thus,

$$\boxed{\frac{d}{ds}\frac{1}{(s^4 - s + 1)^{3/4}} = -\frac{3}{4}(4s^3 - 1)(s^4 - s + 1)^{-7/4}.}$$

Example Notes: As you read the prepared examples, and do the exercises, be sure to verbalize the formulas as you use them. Usually, I reference the verbal versions of the formulas. The *Generalized Power*

Rule can be spoken as "the derivative of a function to a power is that power times the base function raised to one less power, times the derivative of the base function." This is what I recited to myself as I typed in equation (S-2). ■

Example 8.6. ■

8.7. The problem: $\dfrac{d}{dx}x^3\sqrt{1+3x^2}$. This is the derivative of a product of two functions. I'll use the Product Rule.

$$\begin{aligned}\frac{d}{dx}x^3\sqrt{1+3x^2} &= x^3\frac{d}{dx}(1+3x^2)^{1/2} + (1+3x^2)^{1/2}\frac{d\,x^3}{dx}\\ &= \frac{1}{2}x^3(1+3x^2)^{-1/2}\frac{d}{dx}(1+3x^2) + (1+3x^2)^{1/2}(3x^2)\\ &= \frac{1}{2}x^3(1+3x^2)^{-1/2}(6x) + (1+3x^2)^{1/2}(3x^2)\end{aligned}$$

This finishes the calculus part of the problem. Above, I have applied the power rule (*verbalize!*) several times. We now start the algebra part of the problem.

$$\begin{aligned}\frac{d}{dx}x^3\sqrt{1+3x^2} &= \frac{d}{dx}x^3(1+3x^2)^{1/2}\\ &= x^3(\frac{1}{2})(1+3x^2)^{-1/2}(6x) + (1+3x^2)^{1/2}(3x^2)\\ &= 3x^2(1+3x^2)^{-1/2}(x^2+(1+3x^2))\\ &= 3x^2(1+3x^2)^{-1/2}(4x^2+1)\end{aligned}$$

Study the algebra steps — make sure these are familiar to you.

Thus,

$$\boxed{\frac{d}{dx} x^3 \sqrt{1+3x^2} = \frac{3x^2(4x^2+1)}{\sqrt{1+3x^2}}.}$$

Example 8.7. ■

8.8. We have initially the derivative of a quotient. Use the Quotient Rule.

$$\frac{d}{dw}\frac{w\sqrt{w}}{(3w^3+1)^6} = \frac{d}{dw}\frac{w^{3/2}}{(3w^3+1)^6}$$
$$= \frac{(3w^3+1)^6(3/2)w^{1/2} - w^{3/2}(6)(3w^3+1)^5(9w^2)}{(3w^3+1)^{12}} \tag{S-3}$$

This finishes the calculus step. In the above calculation, I used the Power Rule to calculate

$$\frac{d}{dw}(3w^3+1)^6 = 6(3w^3+1)^5\frac{d}{dw}(3w^3+1) = 6(3w^3+1)^5(9w^2)$$

I'll leave the algebra for you to verify. My simplified answer is

$$\boxed{\frac{d}{dw}\frac{w\sqrt{w}}{(3w^3+1)^6} = \frac{3\sqrt{w}(1-33w^3)}{2(3w^3+1)^7}} \tag{S-4}$$

Keep working on (S-3) until you can obtain (S-4). Example 8.8. ■

9.1. We are simply asked to calculate

$$\left.\frac{d}{dx}\sin(x)\right|_{x=\pi/3}.$$

From formula (1) of Theorem 9.1,

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Therefore,

$$\begin{aligned}\left.\frac{d}{dx}\sin(x)\right|_{x=\pi/3} &= \cos(x)|_{x=\pi/3} \\ &= \cos(\pi/3) \\ &= \frac{1}{2}. \qquad \text{(S-5)}\end{aligned}$$

Now turn to the problem of finding the equation of the tangent line at $x = \pi/3$. This is just *straight-line knowledge*. We need two pieces of information: A point on the target line, and the slope of the target line.

The Point: $y = \sin(x)$, when $x = \pi/3$, $y = \sin(x) = \sin(\pi/3) = \sqrt{3}/2$. The given point then is

$$P(\frac{\pi}{3}, \frac{\sqrt{3}}{2}).$$

The Slope: Our target line is tangent to the graph of $y = \sin(x)$ at the point $x = \pi/3$. This is the interpretation of the derivative. Thus

$$m_{\tan} = \frac{dy}{dx}\bigg|_{x=\pi/3} = \frac{1}{2},$$

the last equality due to our earlier computation (S-5).

Equation of Tangent Line: $y - y_0 = m(x - x_0)$ is the point-slope form of the equation of a line that passes through $P(x_0, y_0)$ having slope m. Now we plug in the data,

$$\boxed{y - \frac{\sqrt{3}}{2} = \frac{1}{2}(x - \frac{\pi}{3}).}$$

Example 9.1. ■

9.2. The problem: Calculate $\dfrac{d}{dx} x^3 \sin(x)$.

We are asked to differentiate

(a) a sum (b) a product (c) a quotient (d) a sine function

We use standard techniques.

$$\begin{aligned} \frac{d}{dx} x^3 \sin(x) &= x^3 \frac{d}{dx} \sin(x) + \sin(x) \frac{d\, x^3}{dx} \\ &= x^3 \cos(x) + \sin(x)(3x^2) \\ &= x^2 (x \cos(x) + 3 \sin(x)) \end{aligned}$$

That seemed easy. Example 9.2. ■

9.3. We want to calculate the derivative of the cosine of a function of x; use the (2) Trigonometric Derivative Formulas.

$$\begin{aligned}\frac{d}{dx}\cos(3x^4) &= -\sin(3x^4)\frac{d\,3x^4}{dx}\\ &= -\sin(3x^4)(12x^3)\\ &= -12x^3\sin(3x^4)\end{aligned}$$

Example 9.3. ■

10.1. We use the Power Rule to differentiate.

$$f(x) = x^5$$
$$f'(x) = 5x^4$$
$$f''(x) = 20x^3$$

Example 10.1. ∎

10.2. We proceed as follows.

$$
\begin{aligned}
f(x) &= x^5 \\
f'(x) &= 5x^4 \\
f''(x) &= 20x^3 \\
f'''(x) &= 60x^2 \\
f^{(4)}(x) &= 120x \\
f^{(5)}(x) &= 120 \\
f^{(6)}(x) &= 0
\end{aligned}
$$

For this function, beginning with the 6^{th}-derivative, all derivatives are identically 0. (This is characteristic of polynomials.)

$$f^{(n)}(x) = 0 \qquad n \geq 6.$$

Example 10.2. ■

11.1. We follow the steps of implict differentiation.

Step 1: Think of y as an explicit function of x — but don't bother to try to find this explicit function.

Step 2: Take the equation

$$x^2 + y^2 = 1,$$

and differentiate both sides with respect to x,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d\,1}{dx},$$

and utilize the properties of differentiation

$$\begin{aligned} \frac{d\,x^2}{dx} + \frac{d\,y^2}{dx} &= 0 && \triangleleft \text{ Additive Prop.} \\ 2x + 2y\frac{dy}{dx} &= 0. && \triangleleft \text{ Power Rule} \end{aligned} \tag{S-6}$$

Thus,

$$\boxed{2x + 2y\frac{dy}{dx} = 0.} \tag{S-7}$$

In line (S-6), we have utilized Step 1. We treated y as if it were a function of x. The derivative of y^2, where y is a function of x, is, according to the Power Rule, the exponent (2) times the base function raised to one less power (y), times the derivative of the base function (that's dy/dx).

This completes Step 2. You'll note the last equation generated, equation (S-7), is an equation involving the variables x, y, and dy/dx, as was predicted by (9).

Step 3: Solve (S-6) for dy/dx.

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}.} \tag{S-8}$$

What does this representation of dy/dx mean? Notice that the answer is in terms of both x *and* y. If the curve under consideration defined y as a function of x, then knowledge of the value of x characterizes the point on the curve under consideration. If the curve does not define y as a function of x, as is the case for the circle, then knowledge of x does not uniquely characterize the point on the curve. Corresponding to a particular x there is, if fact, *two* points on the curve — each having a different tangent line.

Therefore, it should not be surprising that the representation of dy/dx depends on both x and y. We need to know the complete set of coordinates in order to characterize the point on the curve.

Thus,

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(x,y)=(1/2,\sqrt{3}/2)} &= -\frac{1/2}{\sqrt{3}/2} \\ &= \frac{1}{\sqrt{3}}. \end{aligned}$$

This jives nicely with the much longer calculation (6).

Similarly,

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1/2,-\sqrt{3}/2)} = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}.$$

This corresponds to (7) made earlier.

Example 11.1. ■

11.2. Be simple minded, follow the procedure.

$$x^{10} - 2xy - y^{10} = 0 \tag{S-9}$$

$$\frac{d}{dx}(x^{10} - 2xy - y^{10}) = \frac{d\,0}{dx}$$

$$10x^9 - 2\frac{d\,xy}{dx} - \frac{d\,y^{10}}{dx} = 0$$

Now the second term is a *product* of two functions of x (Step 1, remember?), and the third term is a function of x raised to a power (Step 1, remember?). For the second term we apply the Product Rule, and for the third term, we apply the Power Rule. Continuing now,

$$10x^9 - 2\frac{d\,xy}{dx} - \frac{d\,y^{10}}{dx} = 0$$

$$10x^9 - 2(x\frac{dy}{dx} + y\frac{dx}{dx}) - 10y^9\frac{dy}{dx} = 0$$

$$10x^9 - 2x\frac{dy}{dx} - 2y - 10y^9\frac{dy}{dx} = 0.$$

This is the end of Step 2. We now go into Step 3: Solve for dy/dx

$$\begin{aligned} 10x^9 - 2x\frac{dy}{dx} - 2y - 10y^9\frac{dy}{dx} &= 0 \\ (10y^9 + 2x)\frac{dy}{dx} &= 10x^9 - 2y \\ (5y^9 + x)\frac{dy}{dx} &= 5x^9 - y. \end{aligned}$$

Finally,

$$\boxed{\frac{dy}{dx} = \frac{5x^9 - y}{5y^9 + x}.}$$

For a particular numerical calculation, notice that the point

$$(\, x_0, y_0 \,) = (\, 3^{9/80}, 3^{1/80} \,) \tag{S-10}$$

satisfies the defining equation (S-9) (it took me awhile to find this point, by the way!). The slope of the line tangent to the graph of (S-9) at the point (S-10) is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(x_0,y_0)} &= \frac{5x_0^9 - y_0}{5y_0^9 + x_0} \\ &= \frac{5\,3^{81/80} - 3^{1/80}}{5\,3^{9/80} + 3^{9/80}} \\ &= \frac{7}{3}\,\frac{1}{3^{1/10}} = \frac{7}{9}3^{9/10} \\ &\approx 2.090569740 \end{aligned}$$

Please verify the algebraic trench warfare above. Example 11.2. ■

11.3. Given are given an equation $x \sin(xy) = 1$ involving x and y. It would be difficult, but not impossible, to solve for y; it is entirely too much trouble. I'll use implicit differentiation.

We begin by thinking (Step 1) of the symbol y as an anonymous function of x, then we differentiation both sides of the equation as per Step 2

$$\begin{aligned} x \sin(xy) &= 1 \\ \frac{d}{dx} x \sin(xy) &= \frac{d\,1}{dx} \\ x \frac{d}{dx} \sin(xy) + \sin(xy) &= 0. \end{aligned}$$

Now, in the first term, we have the derivative of the sine of some function of x we'll use Trig. (1),

$$\begin{aligned} x \frac{d}{dx} \sin(xy) + \sin(xy) &= 0 \\ x(\cos(xy) \frac{d}{dx} xy) + \sin(xy) &= 0. \end{aligned}$$

Now we have a product of two functions of x; use the Product Rule,

$$x(\cos(xy)\frac{d}{dx}xy) + \sin(xy) = 0$$
$$x\cos(xy)(x\frac{dy}{dx} + y) + \sin(xy) = 0.$$

This finishes Step 2. Now we solve for dy/dx.

$$x\cos(xy)(x\frac{dy}{dx} + y) + \sin(xy) = 0$$
$$x^2\cos(xy)\frac{dy}{dx} + xy\cos(xy) = -\sin(xy)$$
$$x^2\cos(xy)\frac{dy}{dx} = -\sin(xy) - xy\cos(xy)$$
$$\frac{dy}{dx} = -\frac{\sin(xy) + xy\cos(xy)}{x^2\cos(xy)}.$$

Thus,

$$\boxed{\frac{dy}{dx} = -\frac{\sin(xy) + xy\cos(xy)}{x^2\cos(xy)}}$$

That's the end of Step 3 and of this example. Example 11.3. ■

11.4. We have calculate dy/dx already, Example 11.1, so I'll skip over that part:

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{S-11}$$

Second Derivative: The second derivative with respect to x is the derivative of dy/dx with respect to x, i.e.,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}.$$

We need only differentiation both sides of (S-11) with respect to x:

$$\frac{d}{dx}\frac{dy}{dx} = -\frac{d}{dx}\frac{x}{y}. \tag{S-12}$$

The left-hand side of (S-12) is the second derivative, the right-hand side is the derivative of the quotient of two functions of x — yes, we take the implicit function attitude that y is an (implicit) function of x.

Let $y' = dy/dx$ for simplicity of notation (and typing). Continuing the calculation from (S-12),

$$\begin{aligned}\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} &= -\frac{d}{dx}\frac{x}{y} \\ &= -\frac{y(1) - xy'}{y^2} \\ &= -\frac{y - xy'}{y^2}.\end{aligned}$$

This then is the desired second derivative. Notice that

$$y'' = -\frac{y - xy'}{y^2}$$

is represented in terms of x, y, and y'. The answer can be improved on by substituting our earlier calculation for y'.

$$
\begin{aligned}
y'' &= -\frac{y - xy'}{y^2} \\
&= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} && \triangleleft \text{ from (S-11)} \\
&= -\frac{y^2 + x^2}{y^3} && \triangleleft \text{ algebra!} \\
&= -\frac{1}{y^3}, && \text{(S-13)}
\end{aligned}
$$

where, in the last step, we simplified $y^2 + x^2$ down to 1, since it is understood that the (x, y) pair satisfies the underlying equation $x^2 + y^2 = 1$. Thus,

$$\boxed{y'' = -\frac{1}{y^3}.} \tag{S-14}$$

Third Derivative: To calculate the third derivative, we take the derivative of the second derivative, (S-14):

$$
\begin{aligned}
y''' = \frac{dy''}{dx} &= -\frac{d}{dx}y^{-3} && \triangleleft \text{ from (S-14)} \\
&= 3y^{-4}\frac{dy}{dx} && \triangleleft \text{ Power Rule} \\
&= 3y^{-4}\left(-\frac{x}{y}\right) && \triangleleft \text{ from (S-11)} \\
&= -\frac{3x}{y^5}
\end{aligned}
$$

Thus,

$$\boxed{y''' = -\frac{3x}{y^5}}$$

Example 11.4. ■

12.1. We simply follow the procedure for finding the absolute extrema. First note that f is continuous over the interval $[-1, 2]$ so the procedure applies.

Calculate Critical Points: Since f is a polynomial, it is everywhere differentiable. This means the only critical points are the ones where $f' = 0$.

$$f(x) = 3 + 4x - 3x^3$$
$$f'(x) = 4 - 9x^2$$

Now, set $f'(x) = 0$ and solve for x:

$$4 - 9x^2 = 0$$
$$9x^2 = 4$$
$$x^2 = \frac{4}{9}$$
$$x = \pm\frac{2}{3}$$

These are the critical points:

$$x_1 = -\frac{2}{3}, \qquad x_2 = \frac{2}{3}.$$

Note that both of these points belong to the target interval $[-1, 2]$. We are not interested in anything outside this interval — had one or more of the numbers been outside the interval, we would have deleted them from our list.

Now we include the endpoints of the interval -1 and 2, and create a table of calculations.

x	$f(x)$	
-1	2	
$-\frac{2}{3}$	$\frac{11}{9}$	
$\frac{2}{3}$	$\frac{43}{9}$	$\Longleftarrow \max_{-1\le x\le 2} f(x)$
2	-13	$\Longleftarrow \min_{-1\le x\le 2} f(x)$

Example 12.1. ■

12.2. Suppose we are at the top of a hill in winter time wearing our skis. The hill is a gentle one, flowing gently downward (as most hills do) towards the bottom. Suddenly and gracefully, we push off and ski down the hill to the bottom. As we slide down, our skis are always *tangent* to the hill. In the course of our mad dash down the hill, we can observe a mathematical truth. (Did you pick up on the keyword *tangent*? I'm sure you did.) Let me try to describe to you this mathematical truth to which I just referred.

Look at the hill from the side view. The hill then looks like a curve in the xy-plane. Let T denote the point at the top of the hill the skier started at, and let B denote the point at the bottom of the hill the skier finished. Imagine drawing a line through these two points; call it the line segment, TB, the gradient of the hill. Now here's the trivial observation that is stated formally by the MEAN VALUE THEOREM: As the skier slides down the hill (remember, the skies are always tangent to the hill), at some point on the hill, the *skiier's skis will be parallel to the gradient of the hill.*

That is the observation: The skiers skies will be parallel to the gradient of the hill at some time in the course of the ski trip. Example 12.2. ■

12.3. Let x_1, $x_2 \in \mathbb{R}$ such that $x_1 < x_2$. Then

$$f(x_1) = x_1 < x_2 = f(x_2).$$

This means that f is strictly increasing. Example 12.3. ■

12.4. Let's look at this function over separate intervals.

On the interval $(0, \infty)$: Let $0 < x_1 < x_2$. We want to argue that $f(x_1) < f(x_2)$. To that end

$$\begin{aligned} f(x_2) - f(x_1) &= x_2^2 - x_1^2 \\ &= (x_2 - x_1)(x_2 + x_1) \\ &> 0 \end{aligned} \tag{S-15}$$

Since $x_1 < x_2$ we know $x_2 - x_1 > 0$. Since $x_1 > 0$ and $x_2 > 0$ we see that $x_1 + x_2 > 0$. These observations justify (S-15).

But $f(x_2) - f(x_1) > 0$ means $f(x_1) < f(x_2)$, which is what we wanted to prove. Thus, f is strictly increasing over the interval $(0, \infty)$.

On the interval $(-\infty, 0)$: Let $x_1 < x_2 < 0$. Prove that $f(x_1) > f(x_1)$. Indeed, as before,

$$\begin{aligned} f(x_2) - f(x_1) &= x_2^2 - x_1^2 \\ &= (x_2 - x_1)(x_2 + x_1) \\ &< 0 \end{aligned} \tag{S-16}$$

Since $x_1 < x_2$ we know $x_2 - x_1 > 0$. Since $x_1 < 0$ and $x_2 < 0$ we see that $x_1 + x_2 < 0$. These observations justify (S-16). (Recall: the product of a positive number and a negative number is a negative number!)

But $f(x_2) - f(x_1) < 0$ means $f(x_1) > f(x_2)$, which is what we wanted to prove. Thus, f is strictly decreasing over the interval $(\,-\infty, 0\,)$.

Example 12.4. ■

12.5. We simple take the first derivative of f

$$f'(x) = 4x^3 - 16x. \tag{S-17}$$

We ask ourselves the question: Where is f' positive and where is f' negative? To answer that question, we simply use standard techniques.

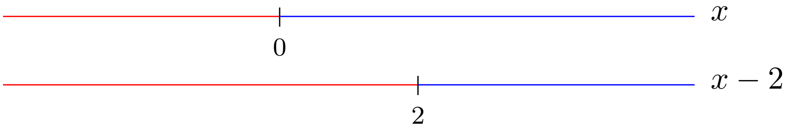
Begin by completely factoring the expression (S-17):

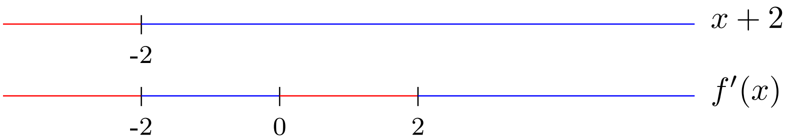
$$f'(x) = 4x(x^2 - 4) = 4x(x - 2)(x + 2), \tag{S-18}$$

where you can see that I have taken it easy on myself in the choice of the function.

Next use the *Sign Chart Method* for analyzing when f' is positive and negative.

The *Sign Chart* of $f'(x) = 4x(x - 2)(x + 2)$





legend: ● negative $(-)$ ● positive $(+)$

Intervals of increase are indicated in blue while intervals of decrease are represented in red. The table of summary results appears to the right.

Table of Increase/Decrease of f

f is decreasing on:	$(-\infty, -2)$
f is increasing on:	$(-2, 0)$
f is decreasing on:	$(0, 2)$
f is increasing on:	$(2, +\infty)$

Example 12.5. ■

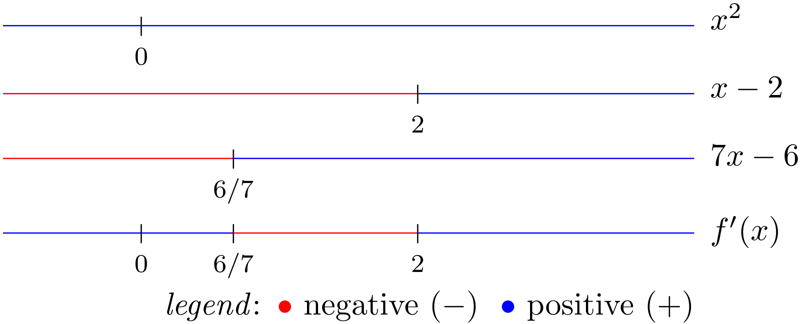
12.6. First calculate the first derivative and completely factor the result.

$$\begin{aligned} f'(x) &= 4x^3(x-2)^3 + 3x^2(x-2)^4 && \triangleleft \text{ Power Rule} \\ &= x^2(x-2)^3[4x+3(x-2)] && \triangleleft \text{ factor it} \\ &= x^2(x-2)^3(7x-6) && \triangleleft \text{ combine similar terms} \end{aligned}$$

Thus,

$$f'(x) = x^2(x-2)^3(7x-6).$$

The *Sign Chart* of $f'(x) = x^2(x-2)^3(7x-6)$



Based on the *Sign Chart* results, it is now easy to locate the intervals of increase and decrease. The blue indicates intervals of increase and the red represents intervals of decrease. A table of results appears to the right.

The Table of Increase/Decrease of f

f is increaseing on:	$(-\infty, 0)$
f is increasing on:	$(0, 6/7)$
f is decreasing on:	$(6/7, 2)$
f is increasing on:	$(2, +\infty)$

Example Notes: The sign of the factor x^2 did not change at $x = 0$; consequently, the monotone behavior did not change at $x = 0$. Actually, in the above table we could have combined the first two intervals and simply stated that "f is increasing on the interval $(-\infty, 6/7)$."

▪ What goes on at $x = 0$? This point is called a *saddle point* is a the topic of discussion later in these notes.

▪ The second factor of f' is $(x-2)^3$; however in the sign chart I did not include the exponent. This was only for convenience. Since the exponent is an odd integer the sign of $(x-2)^3$ is always the same as the sign of $x-2$, in the spirit of simplification, I analyzed the sign of $x-2$.

■ For the first factor, I did include the exponent. This is because the fact that the exponent is even effects the sign of the factor—causing it to be always a nonnegative factor. ■

Example 12.6. ■

12.7. The first derivative of f is given by

$$f'(x) = 4x(x^2 - 4) = 4x(x-2)(x+2). \qquad \text{(S-19)}$$

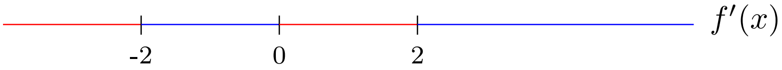
To find the critical points, we look for points at which f' does not exist (there are none) and points at which $f'(x) = 0$. No fancy solving methods are necessary in (S-19) because the expression is completely factored. It is clear form (S-19) that $f'(x) = 0$ with $x = 0, -2, 2$:

$$\text{Critical Points: } 0, -2, 2.$$

Now, we must examine the first derivative to the left and right of these numbers. There is a couple of ways you can do this Examine the *Sign Chart* of f', or examine *test points*.

Sign Chart Method: From Example 12.5 the *Sign Chart of f'* was computed to be

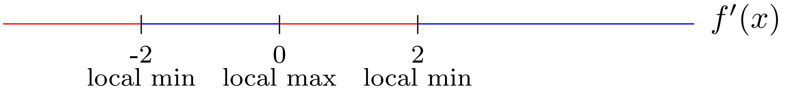
The *Sign Chart* of $f'(x) = 4x(x-2)(x+2)$



legend: • negative (−) • positive (+)

In the case of $x = -2$, the *Sign Chart* indicates that f' is negative to the left and positive to the right of $x = -2$. By Theorem 12.12, f has a local minimum there. For $x = 0$, the *Sign Chart* shows that f' is positive to the left of $x = 0$ and negative to the right; this means that $x = 0$ is a local maximum. Finally, to the left of $x = 2$, f' is negative and to the right f' is positive.

The *Sign Chart* of $f'(x) = 4x(x-2)(x+2)$



legend: ● negative $(-)$ ● positive $(+)$

Test Points: The creation of a *Sign Chart* does take time. Another quicker method is to try some test points. Let me illustrated this method for the critical point $x = -2$.

Take a number slightly to the left and slightly to the right of $x = -2$.

$$f'(-2.1) < 0$$

$$f'(-1.9) > 0$$

We see from these calculations that f' is negative slightly to the left of -2 and positive slightly to the right of -2. This indicates, by Theorem 12.12, that f has a local minimum at $x = -2$.

Caveat Care must be taken not to choose a test point too far form the critical point. You can see from the *Sign Chart* that the sign of f' changes several times. Should you choose a test point far enough away from the critical point, the sign of f' may change and consequently, leading you to a “false” conclusion.

Example 12.7. ■

12.8. The first derivative is given by

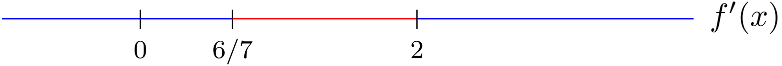
$$f'(x) = x^2(x-2)^3(7x-6).$$

It is clear from this completely factored form of the first derivative that the critical points are

$$\text{Critical Points: } 0, \frac{6}{7}, 2.$$

The *Sign Chart* of f', as computed in EXAMPLE 12.6 is

The *Sign Chart* of $f'(x) = x^2(x-2)^3(7x-6)$



legend: • negative (−) • positive (+)

Notice that f' is positive to the left of $x = 0$ and positive to the right of $x = 0$. This means that f is increases up to the critical point, levels off to a horizontal tangent at $x = 0$ (since $f'(0) = 0$) then continues to increase to the right of $x = 0$. Thus, f' does not change sign at

$x = 0$; this means, from Theorem 12.12(3), f has a (increasing) *saddle point* at $x = 0$.

The other two critical points are $x = 6/7$ and $x = 2$. The function f has a local maximum at $x = 6/7$ and has a local mimimum at $x = 2$. A table of summary results is given to the right.

Classification of Critical Points

0	saddle point
6/7	local maximum
2	local mimimum

Example 12.8. ■

Important Points

To begin with, by the EXTREME VALUE THEOREM, what we are look for exists — we just have to find them.

Let $x_{\min}$ denote a value of x at which f attains it's absolute minimum, i.e.

$$f(x_{\min}) = \min_{a \le x \le b} f(x).$$

We need to "find" $x_{\min}$. Where can $x_{\min}$ be? Logically speaking, $x_{\min}$ is either an endpoint, a point at which f' does not exist, or a point at which f' does exist. This represents an exhaustive analysis of $x_{\min}$.

Now by FERMAT'S THEOREM, if f' exists at $x_{\min}$, then $f'(x_{\min}) = 0$ — this is because $x_{\min}$ is a local extrema (since it is an absolute extrema).

To update our analysis: Where can $x_{\min}$ be? It can be an endpoint, or a point where f' does not exist, or at a point at which $f' = 0$. Or, in other words, $x_{\min}$ is either an endpoint or a critical point.

Therefore, if we list all the endpoints and critical points, $x_{\min}$ **must** be listed among them. Which one is it? Since f has an absolute minimum at x_{min}, then the smallest of the numbers in the right-hand column of the table **must** be the absolute minimum.

A similar reasoning for the absolute maximum. Important Point ■

A critical point is either a local minimum, a local maximum, or a saddle point. The phrase *classifying critical points* refers to, in this context, the problem of determining whether a given critical point is either a local minimum, local maximum, or a saddle point.

Important Point ▪

THEOREM 12.7 would not be true otherwise!

Example. Define a function f by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}$$

Think of the interval $[a, b]$ as $[0, 1]$. The above function is differentiable on the open interval $(0, 1)$, in fact, $f'(x) = 1$ there, but not continuous at $x = 0$.

Let's calculate the right-hand side of equation (3):

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = 1 - 1 = 0$$

Yet, according to the MEAN VALUE THEOREM, there should be some number c, $0 < c < 1$, such that $f'(c) = 0$. (Recall, 0 is the value of the right-hand side of equation (3).) But f' is never equal to 0 since $f'(x) = 1$ for all $x \in (0, 1)$. Thus the conclusion of the MEAN VALUE THEOREM *is not valid* for this function.

If the function f is not required to be continuous at the endpoints, one could assign any arbitrary values to f at the endpoint; consequently, there would be no reason to expect equation (3) to be true.

For those who want to know more. One could ask, "Why split hairs? Why not just require f to be differentiable over $[a, b]$?" We could certainly do that, but that would reduce the number of functions to which the theorem can be applied. For example, the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin(1/x) & \text{if } x \neq 0. \end{cases} \tag{I-1}$$

Think of the interval $[a, b]$ as $[0, 1]$ again. This function is continuous on $[0, 1]$ and differentiable on $(0, 1)$, but f is *not differentiable* at $x = 0$. (By not differentiable at the endpoint $x = 0$, I mean the right-hand derivative of f at 0 does not exist.)

The Mean Value Theorem is applicable to this function, but if we substitute the above revised wording, the "new" Mean Value Theorem does not apply ... because this f is not differentiable on $[0, 1]$.

Exercise Verify the properties of the function f defined in (I-1): Verify that f is continuous at $x = 0$ but *not* differentiable at $x = 0$. (Away from zero, the function f looks like $x \sin(1/x)$—this function is differentiable and the usual rules of differentiation can be applied to compute its derivative. ■

Important Point ■

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Page numbers in boldface indicate the definitive source of information about the item.