THE UNIVERSITY OF AKRON
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calculus menu

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Directory

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Differentiation

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1. Introduction

Prerequisite: Limits, Continuity.

2. Motivating the concept

2.1. Introduction

The term *differentiability* concerns the study of the limit,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{1}$$

This limit insinuates itself into a variety of physical settings. In the following sections, we introduce several situations in which (1) arises "naturally."

2.2. Instantaneous Velocity

Short Summary: Let P be a particle moving along an axis of real numbers and let this axis be called the s-axis. At any given time t, the particle P holds a unique position on the s-axis. The position of the particle is then characterized by the value on the number scale of the point at which P resides at the given time. This relationship between time, t, and the position of the particle on the s-axis defines a functional relationship.

Define a function to by

$$s = f(t) = \text{The position of the particle } P \text{ at time } t.$$

At any time t, the *instantaneous velocity* of the particle P is given by

$$v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}. \tag{2}$$

Thus, instantaneous velocity, (2), is defined by kind of limit given in (1).

For a more detailed discussion of this topic, Click here.

2.3. Tangent to a Curve

Short Summary. In this section we discuss the *Fundamental Problem of Differential Calculus*: Given a function, $y = f(x)$, and a point, $P(\,a, f(a)\,)$, on the graph of the function, define and calculate the equation of the line tangent to the graph at the given point. It turns out that the *slope*, $m_{\,\text{tan}}$, of the line tangent to the graph at the point $P(\,a, f(a)\,)$ is given by

$$m_{\,\text{tan}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{3}$$

But this limit is precisely the same kind of limit given in (1).

Click here to learn more (including some interesting graphics).

2.4. Rate of Change

Short Summary. In many applied settings we have two competing variables x and y that are related $(y = f(x))$, and we are interested in how changes in one variable, x say, affect changes in the other variable, y. In particular, we are interested in measuring how fast y changes with respect to unit changes in the x. This *rate of change* of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{4}$$

Click here for a more verbose discussion of this topic.

3. The Definition of Derivative

In this section we formally define the derivative of a function and develop some of the very important mechanical skills.

3.1. The Derivative Defined

A definition is usually the jumping off point of any mathematical study, and we are not different.

Definition 3.1. Let f be a function and $a \in \text{Dom}(f)$ such that f is defined in an open interval containing a. We say that f is *differentiable* at $x = a$ provided

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.} \tag{1}$$

In this case, define the *derivative*, $f'(a)$, to be

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}} \tag{2}$$

Definition Notes: Think of the requirement that f be defined in an open interval containing a as necessary for the two-sided limit (1) to exist.

- **Difference Quotient.** The ratio

$$\frac{f(a+h)-f(a)}{h} \tag{3}$$

is called the *difference quotient*. Within the context of applications (see velocity, tangency, and rate of change), the difference quotient itself has physical or geometric interpretation. The *student is encouraged* to read the above referenced articles for more detail.

- The definition, as stated can be rewritten using other variables. Let $x = a + h$, then $h = x - a$. Now, intuitively, $h \to 0$ is equivalent to $x \to a$ since $x = a + h$, and $h \to 0$. Thus, (2) is sometimes written as

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \tag{4}$$

You guessed it, the ratio

$$\frac{f(x) - f(a)}{x - a}$$

is called the *difference quotient*. ■

Interpretations: In the section Motivating the Concept, we introduced differentiation indirectly, through three applications: Velocity, Tangent Line, and Rate of Change. Throughout these notes, the second interpretation of derivative will be utilized to illustrate the concepts.

Let $y = f(x)$ be a function and $a \in \text{Dom}(f)$ such that $f'(a)$ exists. Then $f'(a)$ is interpreted as the *slope of the line tangent* to graph of f at the point $(\,a, f(a)\,)$. The equation of the tangent line is given by

$$y = f(a) + f'(a)(x - a).$$

EXAMPLE 3.1. (Skill Level 0) Let $f(x) = 3x^2$, and $a = 1$.

a. Find the difference quotient of f.

b. Find $f'(a)$.

The next example is meant to verify the assertion in the highlighted box concerning the equation of the tangent line.

EXAMPLE 3.2. If f has a derivative at $x = a$, show that the equation for the line tangent to the graph of f at $x = a$ is given by

$$y = f(a) + f'(a)(x - a). \tag{5}$$

EXAMPLE 3.3. (Continued from EXAMPLE 3.1) For $f(x) = 3x^2$, find the equation of the line tangent to the graph of f at the point on the graph corresponding to $x = 1$.

Strategy: Let's plot some strategy on how to evaluate the limit (2) in the definition of derivative. As h goes to 0, both the numerator *and* denominator go to 0. You'll recall my famous *Empirical Observation*. This is still our guide for reasoning and simplifying the *difference quotient*.

Let's specialize the *Empirical Observation* to limits of the difference quotient.

Differentiation Strategy.
When trying to evaluate the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

you must develop, through algebraic manipulation, a factor of h in the numerator. The factor of h in the numerator and h in the denominator can and must be cancelled. Hopefully, the limit can then be evaluated.

This *Differentiation Strategy* can generally be carried out when the function f is an algebraic function. When we take up the topic of differentiating trigonometric functions, the problem is not so simple as described in the *Strategy*. Later, then *exponential*, *logarithmic*, and other types of more complex functions are studied, the advice about factoring and cancelling out the 'h' is no longer practical.

Exercise 3.1. Consider the function $f(x) = 5x^2 - 2$. Keeping in mind the Differentiation Strategy, and the techniques of Example 3.1,

a. Find the difference quotient at $a = 1$.
b. Calculate $f'(a)$.
c. Find the equation of the line tangent to the graph of f at $x = 1$.

If you're are a beginner, take out a sheet of paper and write out my solution to Example 3.1 and line by line, make appropriate changes in the argument — here, we have a slightly different function.

In the next two sections, we present more sample calculations of derivatives.

3.2. The Derivative as a Numerical Value

In this section we exhibit additional examples and techniques. Oh, there are some exercises for you too so you won't feel left out.

Example 3.4. (A Sample Calculation) Find the derivative of $f(x) = x^2$ at $x = 2$.

Example 3.5. (Calculation of Tangent Line) Consider the function $f(x) = x^2$, find the equation of the line tangent to the graph of f at the point corresponding to $x = 2$.

Example 3.6. (Skill Level 1) Consider the function $f(x) = \sqrt{2x}$. Find $f'(3)$

In the upcoming section Some Basic Differentiation Rules, we obtain rules for finding derivatives without invoking the limit definition of derivative. The basic definition still must hold the predominate position. If you encounter a function not covered by the rules (admittedly, this will be very infrequent until you advance to a high level in your studies), you will have to resort to the definition to evaluate derivatives.

The next example is provided for your additional education. We want to differentiate a more pathological function. Read it if you dare.

Example 3.7. (Skill Level 5) Define the function

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number} \\ 0 & \text{otherwise} \end{cases}$$

Show that $f'(0)$ exists, and $f'(0) = 0$.

Summary: Before we leave, let's summarize the major points to date.

▪ The derivative of a function f at a particular point $x = a$ is given by

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.} \tag{6}$$

▪ If a is a numerical value (as opposed to a symbolic quantity) then $f'(a)$ is a *numerical value*. The interpretation of this numerical value is two-fold: (1) it is the slope of the line *tangent* to the graph of f at $x = a$. See Example 3.5. And (2), $f'(a)$ is a measure of the rate of change of the dependent variable with respect to the independent variable. More on this later.

3.3. The Derivative as a Function

One gets a little tired of calculating the derivative as a numerical value. If we calculate a derivative a one value of x, then need the derivative at another value of x, we must rework the (2) all over again, sigh!

In Definition 3.1, if we allow a to be a mathematical variable then we may look upon f' as a function. In the definition just cited, replace the letter a with the more traditional symbol for a mathematical variable x to get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{7}$$

Now, f' is referred to the *derivative* of f.

Domain Analysis: The domain of the derivative f' of f is

$$\mathrm{Dom}(f') = \{\, x \in \mathrm{Dom}(f) \mid \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exits} \,\}.$$

In general, $\mathrm{Dom}(f') \neq \mathrm{Dom}(f)$, but of course $\mathrm{Dom}(f') \subseteq \mathrm{Dom}(f)$.

For a given value $a \in \mathrm{Dom}(f')$, the numerical value $f'(a)$ represents derivative information only about a, whereas, the derivative *function* contains derivative information about all $a \in \mathrm{Dom}(f')$.

EXAMPLE 3.8. Consider the function $f(x) = x^2$. Show that the derivative f' of f is given by $f'(x) = 2x$.

EXERCISE 3.2. Consider the function $f(x) = 5x^2 - 2$. Show that the derivative function is given by $f'(x) = 10x$. (Hint: Follow the pattern outline in the solution to EXAMPLE 3.8.)

EXAMPLE 3.9. (Skill Level 1) Consider the function $f(x) = \sqrt{2x}$. Find the derivative function, f'.

EXERCISE 3.3. For $f(x) = \sqrt{3x}$, find $f'(x)$.

EXERCISE 3.4. Here is a little trickier one. For $f(x) = \dfrac{1}{\sqrt{x}}$, find $f'(x)$.

3.4. Continuity and Differentiation Related

Below we state the relationship between the function properties of continuity and differentiability.

Theorem 3.2. *Let f be a differentiable function at $a \in \mathrm{Dom}(f)$. Then, f is continuous at $a \in \mathrm{Dom}(f)$.*

Proof. Let $f'(a)$ denote the derivative of f at $x = a$. Then using the alternate definition of derivative given in equation (4), we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{8}$$

We want to prove

$$\lim_{x \to a} f(x) = f(a). \tag{9}$$

We want to relate what we know, (8), with what we want to know (prove), (9). The relationship is

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a). \tag{10}$$

Therefore,

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(f(a) + \frac{f(x)-f(a)}{x-a}(x-a)\right)\\ &= f(a) + \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right)\\ &= f(a) + \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a)\\ &= f(a) + f'(a)(0)\\ &= f(a).\end{aligned}$$

We have used, in several of the lines, standards facts concerning the *Algebra of Limits*. □

The next statement is a logical equivalent to Theorem 3.2 and is sometimes useful in arguing that a given function is not differentiable.

Corollary 3.3. *If f is discontinuous at $x = a$, then f is not differentiable at $x = a$.*

Proof. Suppose f is not continuous at $x = a$. Then either f is differentiable at $x = a$, or f is not differentiable at $x = a$. (The latter is the conclusion we are going for.)

Suppose the contrary, that is, assume that f is differentiable at $x = a$. Then by Theorem 3.2, f is continuous at $x = a$; but this contradicts the initial assumption that we were working a function that is not continuous at $x = a$. Therefore, f cannot be differentiable at $x = a$ (for otherwise, we are faced with contradictory assertions). ☐

EXAMPLE 3.10. Show that the function

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x + 1 & x > 0 \end{cases}$$

is not differentiable at $x = 0$.

3.5. One-Sided Differentiability

Let $f(x)$ be a function. Then $f'(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{11}$$

provided the limit in (11) exists. Now from general theory, a (two-sided) limit exists if and only if both

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \tag{12}$$

and

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \tag{13}$$

exist.

For several reasons, it is convenient to give names to the limits (12) and (13).

Definition 3.4. Let f be defined in an interval I, and let $x \in I$.

(1) Define the *left-hand derivative of* f at x by

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}. \tag{14}$$

provided the limit exists.

(2) Define the *right-hand derivative of* f at x by

$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}. \tag{15}$$

provided the limit exists.

Definition Notes: For the left-hand derivative, the function f needs to be defined in an interval to the left of x. For the right-hand derivative, f must be defined in an interval to the right of x. ■

Based on my remarks preceding the definition, we can now state ...

Theorem 3.5. *Let f be a function defined in an open interval containing x, then $f'(x)$ exists if and only if $f'_-(x)$ and $f'_+(x)$ both exist and $f'_-(x) = f'_+(x)$. In this case,*

$$f'(x) = f'_-(x) = f'_+(x).$$

Theorem Notes: We are simply saying the derivative exists at x if and only if the left-hand derivative equals the right-hand derivative at x.

▪ Turning things around, the theorem states that $f'(x)$ *does not exist* if and only if either $f'_-(x)$ d.n.e., or $f'_+(x)$ d.n.e., or, if they both do exist, $f'_-(x) \neq f'_+(x)$. This statement is logically equivalent to the theorem. It gives a formal way of arguing that a particular derivative does not exist. See below. ▪

Applications: There are two immediate applications to the concept of the one-sided derivative.

1. Use Theorem 3.5 as a device to prove or disprove a function is differentiable.
2. Enables us to have a derivative concept at an endpoint of an interval in which a function is defined.

With regards to (1), the next two examples show how Theorem 3.5 can be used to show a derivative does or does not exist.

Example 3.11. Define a function by $f(x) = |x|$. Prove that f does not have a derivative at $x = 0$.

The next example illustrates how Theorem 3.5 can be used to show a derivative exists.

Example 3.12. Define a function

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Prove the $f'(0)$ exists and $f'(0) = 0$.

3.6. Non differentiability

Discontinuous Functions. One way a function may fail to be differentiable at a particular point is for it not to be continuous there. (See Corollary 3.3.) Example 3.10 was an example of this type of nondifferentiability already encountered.

More interesting examples of nondifferentiability are taken up next. It is "easy" not to have a derivative if the function is not even continuous. If the function is continuous, how can a function avoid having a derivative?

Functions with Corners. A continuous function having a sharp corner will not have a derivative. A function f is said to have a *corner* at $x = a$ provided $f'_-(a)$ and $f'_+(a)$ both exist a finite numbers, but

$$f'_-(a) \neq f'_+(a). \tag{16}$$

Recall, the symbolisms used refer to the left-hand derivative and the right-hand derivative.

An example of a function having a corner is $f(x) = |x|$. Earlier, in Example 3.11, we showed that the $|x|$ function did not have a derivative at $x = 0$ by arguing that $f'_-(0) = -1$ and $f'_+(0) = 1$. Thus, for the function $f(x) = |x|$, (16) is satisfied.

You can create functions with sharp corners by piecing together other functions. For example,

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases} \tag{17}$$

I have pieced two functions together so that where they meet, at $x = 0$, the form a continuous function there.

EXERCISE 3.5. Show that the function defined in (17) is not differentiable at $x = 0$.

Another example, not obtained by piecing functions together, is

$$f(x) = x^{2/3}. \tag{18}$$

EXAMPLE 3.13. Show that the function defined in (18) is not differentiable at $x = 0$.

Functions with Vertical Tangents. Some functions have tangent lines, but the tangent line is vertical. A vertical line either has no slope or has infinite slope depending on your philosophical point of view. Let's agree to say that a function f has a vertical tangent at $x = a$ if either

$$f'_-(a) = f'_+(a) = -\infty$$

or,

$$f'_-(a) = f'_+(a) = +\infty \tag{19}$$

An example of a function having a vertical tangent line is

$$f(x) = x^{1/3}. \tag{20}$$

EXERCISE 3.6. Show that the function in (20) has a vertical tangent at $x = 0$

4. Differentiation Notation

Before we continue with the development of differentiation rules, let's introduce some important notation.

As we are in a primitive state, we don't have a lot of examples to which to refer. Recall that if $f(x) = x^2$, then $f'(x) = 2x$. I'll use this example to illustrate the notations.

4.1. The Prime Notation

This is the notation you have been using. The rule for this notation is

> *Prime Notation Rule*
> When you take the derivative of a function, the name of the derivative function is same as the parent function but with a superscript of a 'prime' ($'$) post-fixed.

Thus the derivative of f is f'. The derivative of g is g', the derivative of h is h'. The derivative of W is W'. Do you get the picture?

Exercise 4.1. What is the notation for the derivative of f'?

Taking this one step further. If f is a function with derivative f', then the variable symbol used for f is also used for f'. Thus, if f is a function of x, so is f'. If g is a function of t, so is g'.

$$\begin{aligned} f(x) &= x^2 & f'(x) &= 2x \\ g(t) &= t^2 & g'(t) &= 2t \\ W(s) &= s^2 & W'(s) &= 2s \\ L'(w) &= w^2 & L''(w) &= 2w \end{aligned}$$

Now, here's the twist. In our discussion on notation, we noted that often, function are defined anonymously — they have no names; in this case, we use the *dependent variable* as the name of the function. Therefore, *The Prime Notation Rule* still holds. For example, if the function is given anonymously, $y = x^2$, then y' is the (anonymous) name of the derivative function: $y' = 2x$.

Area, A, of a square whose sides have length s is given by $A = s^2$. The derivative function is $A' = 2s$.

More examples,

$$\begin{aligned} w &= v^2 & w' &= 2v \\ q' &= x^2 & q'' &= 2x \\ l'' &= y^2 & l''' &= 2y \end{aligned}$$

Numerical Calculations: The prime notation has a problem: It doesn't have an explicit argument. When making numerical evaluations, two approaches are used: force an argument, make up a new notation. Let's illustrate,

Force an argument,

$$y = x^2 \qquad y' = 2x \qquad y'(3) = 6.$$

Make up a new notation,

$$y = x^2 \qquad y' = 2x \qquad y'|_{x=3} = 6.$$

The prime notation has another problem. Some functions has several symbolics in their definition. For example, if we define a function $w = s^2 t^3$, then you might ask, quite rightly, what the independent variable is. Should we consider s is independent variable and t as a constant, or should we consider t the independent variable and s as a constant? This is an important question since the calculation of the derivative depends on what you consider to be the independent variable.

In the next section, we introduce a notation, that, as a built-in feature, tells you what the independent variable is.

4.2. The Leibniz Notation

The Leibniz is a very useful and powerful notation for differentiation. You should make every attempt to understand the notation and utilize it correctly.

Let $y = f(x) = x^2$. The Leibniz notation for the derivative of f is any of the following:

$$\frac{dy}{dx} = 2x \qquad \frac{df}{dx} = 2x.$$

In general, if $y = f(x)$, then we write

$$\frac{dy}{dx} = \text{the derivative of } y \text{ with respect to } x$$

$$\frac{df}{dx} = \text{the derivative of } f \text{ with respect to } x$$

The notation contains within it the name of the dependent variable y (or the name of the function f, your option), *and* a specification of the independent variable x. Thus,

$$\frac{dw}{ds} = \text{the derivative of } w \text{ with respect to } s$$

$$\frac{dV}{dr} = \text{the derivative of } V \text{ with respect to } r$$

Here are some quick examples that will look familiar to you.

$$y = x^2 \qquad \frac{dy}{dx} = 2x$$

$$w = t^2 \qquad \frac{dw}{dt} = 2t$$

$$q = l^2 \qquad \frac{dq}{dl} = 2l$$

$$A = s^2 \qquad \frac{dA}{ds} = 2s$$

I hope you see the pattern. This notation will be used throughout the rest of these tutorials and the rest of your *Calculus* course.

For $y = x^2$, the symbol y represents the value x^2. Rather than writing dy/dx, we quite often write

$$\frac{d\,x^2}{dx} = 2x. \tag{1}$$

The translation of (1) is "The derivative of the function x^2, with respect to x is the function $2x$."

Evalulation. The Leibniz notation one of the problems as the prime notation: No explicit argument. Traditionally, we evaluate derivatives utilizing the notation:

$$y = x^2 \qquad \frac{dy}{dx} = 2x \qquad \left.\frac{dy}{dx}\right|_{x=3} = 6.$$

Notes of Origin of the Notation. The Leibniz Notation has origin in how you represent the definition of derivative.

The Δ-convention: Let $y = f(x)$ be a differentiable function. The definition of $f'(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

Leibniz did not use the h notation; instead he used Δx. Think of Δx as representing a small change in the value of x. Replace the letter h by the symbol Δx in (2) to obtain

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \tag{3}$$

The numerator in (3) is the difference of two y-values. Let

$$\Delta y = f(x + \Delta x) - f(x),$$

then (3) becomes

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{4}$$

Leibniz then denoted the limit of the difference quotient

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{5}$$

where,

$$\begin{aligned} \Delta x &= \text{an independent variable} \\ \Delta y &= f(x + \Delta x) - f(x) \end{aligned}$$

Difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

For additional details, read the section Rate of Change, if you haven't done so already.

d/dx as an Operator. The Leibniz notation is sometimes called an operator. The symbol

$$\frac{d}{dx}$$

then represents the *operation* of differentiation. To differentiate a function $y = x^2$, for example, we *apply* the differentiation operator to the function:

$$y = x^2$$

Apply $\dfrac{d}{dx}$ to both sides of the equation,

$$\frac{d}{dx} y = \frac{d}{dx} x^2$$

or,

$$\frac{dy}{dx} = \frac{d\,x^2}{dx}. \tag{6}$$

The left-hand side of (6) becomes a symbol for the derivative of the function, while the right-hand side is a derivative problem of an explicit function of x. Equation (6) then becomes

$$\frac{dy}{dx} = 2x.$$

Upon the evaluation of the right-hand side of (6) yields the proper notation and answer to the problem of finding the derivative of the function $y = x^2$.

As you go through these notes you will see the role of d/dx as an operator quite often. The technique of implicit differentiation utilizes this "operator" point of view extensively.

5. Three Fundamental Interpretations

Now that we have developed the concept of the derivative, studied the notation and worked on the mechanics of computation, it is time to look again at the fundamental interpretations of the derivative. These interpretations will play in important role throughout the rest of the Calculus sequence ... and beyond.

5.1. Tangent Lines

The Tangent Line Problem was one of the problems used to motivate the concept of the limit of a function. Let us review this problem and state its solution in terms of derivative notation.

Problem. Given a function $y = f(x)$ and a point $a \in \mathrm{Dom}(f)$. The problem is to calculate the *slope* of the line tangent to the graph of f at the point $P(\,a, f(a)\,)$.

Solution. The slope of the line tangent to the graph of f at the point $P(a, f(a))$ is given by $f'(a)$, i.e.

$$f'(a) = \begin{array}{l}\text{the slope of the line tangent to the graph}\\ \text{of } f \text{ at the point } P(a, f(a)).\end{array}$$

This information can be parlayed into the *equation of the tangent line*.

Problem. Find the equation of the line *tangent* to the graph of $y = f(x)$ at the point $P(a, f(a))$.

Solution. The equation of the line tangent to the graph of $y = f(x)$ at the point $P(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a).$$

5.2. Velocity

Suppose a particle moves in a straight line. Call this straight line the s-axis, which will be a real number line. At any time t, the particle,

P, holds a certain position, s, on the s-axis. This establishes position, s, as a function of time, t. Symbolically represent this relationship by

$$s = f(t).$$

We have seen in equation (2) of section 2.2, that the instantaneous velocity of the particle at time t is given by

$$v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

A close comparison of this defining equation with the definition of derivative shows that

$$v(t) = \frac{ds}{dt} = f'(t). \tag{1}$$

That is instantaneous velocity is the first derivative of position with respect to time. This last phrase is important.

Instantaneous velocity of a particle is the derivative of its position with respect to time. That is, if

$$s = f(t) = \begin{array}{l}\text{the position of the particle at time } t \text{ on}\\ \text{the } s\text{-axis, an axis of real numbers}\end{array}$$

then,

$$v(t) = \frac{ds}{dt} = f'(t).$$

5.3. Rate of Change

One of the "generic" interpretations of the derivative is the *rate of change* interpretation. This topic was extensively discussed in Section 2.4. The reader is asked to review that section as well as the section entitled Rate of Change. In that section, the rate of change interpretation was taken up as a problem that motivates the study of the more fundamental concept of the limit of a function.

Let $y = f(x)$ define y as a function of x. According to the discussion, as summarized in Section 2.4 above, the rate at which y (that is, the values of the function f) change with respect to x is given by (see (4))

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Comparing this expression with the definition of derivative, we see that

$$\frac{dy}{dx} = f'(x) = \text{The (instantaneous) rate at which the } y \text{ variable changes per unit change in the variable } x.$$

6. Some Basic Differentiation Rules

One doesn't want to put in any more time calculating derivatives from the definition than is needed to understand what a derivative is. We'll now turn to a systemic development of come standard differentiation formulas.

6.1. The Derivative of a Constant

Let's begin with the trivial, then move on to the sublime.

Theorem 6.1. *Let $c \in \mathbb{R}$ be a fixed constant. Define the function $f(x) = c$, for $x \in \mathbb{R}$. Then f is differentiable, and*

$$f'(x) = 0 \qquad x \in \mathbb{R}.$$

Proof. The proof consists of a simple calculation.

$$\begin{aligned} f(x) &= c \\ f(x+h) &= c \\ f(x+h) - f(x) &= c - c = 0. \end{aligned}$$

The difference quotient,

$$\frac{f(x+h)-f(x)}{h} = 0 \qquad \text{for all } h.$$

Therefore,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} 0 = 0.$$

We have shown that if $f(x) = c$, then $f'(x) = 0$, for all $x \in \mathbb{R}$. $\square$

Comments: The graph of the constant function, $f(x) = c$ is a horizontal line. At any given point $P(\,x, f(x)\,)$, the slope of the line tangent to the graph of f is 0; therefore, it should not be surprising to you that $f'(x) = 0$, for one of the interpretation of $f'(x)$ is that it is the slope of the line tangent to the graph of f a the point $P(\,x, f(x)\,)$.

In terms of the Leibniz notation, the above theorem can be written more eloquently as

$$\boxed{\frac{d\,c}{dx} = 0} \tag{1}$$

Derivative of a Constant.

The derivative of a constant, c, is zero: $\dfrac{d}{dx}c = 0$

For example, if $f(x) = 5$, then $f'(x) = 0$. In Leibniz notation,

$$\frac{d\,5}{dx} = 0$$

EXERCISE 6.1. Calculate $\left.\dfrac{d\,5}{dx}\right|_{x=123.45}$.

6.2. The Power Rule

A *power function* is a function of the form:

$$(\text{some function})^{\text{constant}}$$

Examples of power functions are elementary ones,

$$y = x^2 \qquad w = s^{-23} \qquad z = \sqrt{q} = q^{1/2},$$

and more complex ones,

$$y = (x^2 + 1)^3 \qquad w = \sin^4(x) \qquad W = \left(\frac{1 + 2x}{\cos(x)} \right)^{17}.$$

In each of these, we have a *base function*, raised to a fixed exponent.

In this section, we learn how to differentiate the elementary ones; that is a function of the form

$$y = x^n.$$

Theorem 6.2. (The Power Rule: Junior Grade) *Consider the function $f(x) = x^n$, for some $n \in \mathbb{N}$, the set of natural numbers. Then*

$$f'(x) = nx^{n-1} \qquad x \in \mathbb{R}. \tag{2}$$

Proof.

Before looking a examples, let's rewrite the Power Rule in Leibniz Notation.

The Power Rule JG.
Let $n \in \mathbb{N}$, i.e. $n = 1, 2, 3, \ldots,$ then

$$\frac{d\,x^n}{dx} = nx^{n-1}.$$

EXAMPLE 6.1. Solve each of the following:

$$\frac{d\,x^4}{dx} \quad \frac{d\,x^{123}}{dx} \quad \frac{d\,s^5}{ds} \quad \frac{d\,w^{10}}{dw}.$$

EXAMPLE 6.2. The proof of the Power Rule is hyper-referenced, in this example, we will exhibit the technique of proof by deriving the formula for a particular case. **Problem.** Derive the derivative, from the definition, of $f(x) = x^4$.

Have you gotten the feeling that our rules are still limited? We can't do a whole lot yet. In the next section we will make considerable progress in that regard.

7. The Algebra of Differentiable Functions

7.1. The Algebraic Structure

We now come to a section of these notes with a familiar title. I have been stressing the *algebraic structure* of many of the concepts we encounter in Calculus. The following theorem I include as a structural statement, the operational versions are stated afterwards.

Theorem 7.1. (The Algebra of Differentiable Functions)*Let f and g be defined in an interval containing $x = a$, and let both f and g be differentiable at $x = a$. Finally let $c \in \mathbb{R}$ be a constant. Then each of the functions are also differentiable at $x = a$ as well:*

$$cf \quad f + g \quad f - g \quad fg \quad \frac{f}{g}, \text{provided } g'(a) \neq 0.$$

Proof.

Theorem Notes: You can see that combining differentiable function through addition, subtraction, multiplication, division all yield functions that are differentiable. Spoken differently, if f and g both have tangent lines at $x = a$, then so does cf, $f \pm g$, fg, and f/g, the latter is true provided $g'(a) \neq 0$. ∎

The natural question we address throughout the rest of this section is how do we make calculations. Given that we know

$$f(a), \quad f'(a), \quad g(a), \quad g'(a), \tag{1}$$

how can we use this information to calculate

$$(cf)'(a), \quad (f+g)'(a), \quad (fg)'(a), \quad (f/g)'(a)?$$

Stay tuned for the answers.

7.2. The Homogeneous Property of Derivative

In this section we study how differentiation interacts with multiplies of functions.

Theorem 7.2. (Homogeneity of the Derivative) *Let $u = f(x)$ be differentiable at $x = a$, then*

$$(cf)'(a) = cf'(a).$$

Proof.

The Leibniz Notation is used most frequently to express these formulas. Let's develop the proper notation.

$$(cf)'(a) = \frac{d(cf)}{dx}$$
$$cf'(a) = c\frac{df}{dx}$$

Equating the two,

$$\frac{d(cf)}{dx} = c\frac{df}{dx}$$

Note that I have cleverly let $u = f(x)$ in the statement of Theorem 7.2. Now if we replace the name f of the function with the dependent variable we get,

$$\frac{d(cu)}{dx} = c\frac{du}{dx}$$

Let's elevate this formula to the stature of a boxed in equation.

Homogeneity of Differentiability.
Let u be a differentiable function of x, then

$$\blacksquare \qquad \frac{d(cu)}{dx} = c\frac{du}{dx}$$

We can now increment the level of difficulty a microscopic bit.

EXAMPLE 7.1. Differentiate each of the following.

$$\frac{d\,4x^6}{dx} \qquad \frac{d\,21s^3}{ds} \qquad \frac{d}{dw}\frac{5}{4}w^4.$$

7.3. The Additive Property of Derivative

This section describes how to calculate the derivative of the sum of two functions.

Theorem 7.3. (The Additive Property)*Let* $u = f(x)$ *and* $v = g(x)$ *be differentiable at* $x = a$. *Then*

$$(f+g)'(a) = f'(a) + g'(a). \tag{2}$$

Proof.

The Leibniz notation is used quite often when manipulating complex functions. Equation (2) may be rewritten as

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}. \tag{3}$$

A variation of (3) utilizes dependent variables to name the functions. Let $u = f(x)$ and $v = g(x)$ then

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \tag{4}$$

The Additive Property of Derivative:
Let u and v be a differentiable functions of x, then

■ $$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

EXAMPLE 7.2. (With Attributions) Calculate $\dfrac{d}{dx}(3x^4 - 6x^5)$.

EXAMPLE 7.3. Calculate $\dfrac{d}{dw} 3\csc(w) - 5\cot(w)$.

Exercise 7.1. Calculate $\dfrac{d}{ds} 4s^5 - \frac{2}{3}s^3$.

Exercise 7.2. Consider the function in the Exercise 7.1, i.e., $w = 4s^5 - (2/3)s^3$.

a. Find all values of s at which the tangent line is horizontal.
b. Find all values of s at which the tangent line is parallel to the line $w = 120s + 3$.

Begin by writing down the derivative, $\dfrac{dw}{ds}$, from the solution of Exercise 7.1. Formulate, in terms of setting up certain equations, how to solve each part — and solve.

7.4. The Product Rule

We explain how to calculate the derivative of the product of two functions.

Theorem 7.4. *Let* $u = f(x)$ *and* $v = g(x)$ *be differentiable at* $x = a$*, then*

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a). \tag{5}$$

Proof.

The Leibniz notation is important to master.

$$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}. \tag{6}$$

Finally, if we denote $u = f(x)$ and $v = g(x)$, then (6) becomes,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \tag{7}$$

Thus,

The Produce Rule:
Let u and v be differentiable functions of x, then

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7.4. (With Attributions) Calculate $\dfrac{d}{dx}x^3(2x^9 - 12)$.

Exercise 7.3. For $y = x^{12}(1 + x^2)$, find $\dfrac{dy}{dx}$ using the Product Rule. Work your answer out first and simplify before looking.

7.5. The Quotient Rule

We learn to calculate the derivative of the quotient of two functions.

Theorem 7.5. *Let $u = f(x)$ and $v = g(x)$ be differentiable at $x = a$, then*

$$(f/g)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{[g'(a)]^2}, \tag{8}$$

provided $g(a) \neq 0$.

Proof.

Designating $u = f(x)$ and $v = g(x)$, in Leibniz notation the Quotient Rule becomes

$$\frac{d}{dx}\frac{u}{v} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{9}$$

provided $v \neq 0$.

The Quotient Rule:
Let u and v be differentiable functions of x, then

$$\frac{d}{dx}\frac{u}{v} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad v \neq 0.$$

EXAMPLE 7.5. (With Attributions) Calculate $\dfrac{d}{dx}\dfrac{3x^3 - 2x}{4 - 5x^2}$.

EXAMPLE 7.6. For $w = \dfrac{s^2}{3s^4 - 2}$, find $\dfrac{d}{ds}w$.

Solve the problem first, use good techniques, and simplify before looking at the answer — please!

EXERCISE 7.4. Calculate $\dfrac{d}{dx}\dfrac{7x^6}{6x^8 - 1}$. (Hint: Quotient Rule.)

7.6. The Extended Power Rule: Junior Grade

With the aid of the Quotient Rule, and a little common sense, we can extend the Power Rule.

Consider a function of the form $y = x^n$, where n is a *negative integer*, $n = -1, -2, -3, \ldots$. We want to calculate the derivative of this function. Before we get started, please meditate on the following statement: If n is a negative integer, $-n$ is a positive integer.

Calculate,

$$\begin{aligned}
\frac{d\,x^n}{dx} &= \frac{d}{dx}\frac{1}{x^{-n}} \\
&= \frac{x^{-n}\dfrac{d\,1}{dx} - (1)\dfrac{d}{dx}\,x^{-n}}{(x^{-n})^2} \qquad \triangleleft \text{ Quot. Rule} \\
&= \frac{0 - (-n)(x^{-n-1})}{x^{-2n}} \qquad (10) \\
&= \frac{nx^{-n-1}}{x^{-2n}}
\end{aligned}$$

$$= nx^{-n-1}x^{2n}$$

$$= nx^{n-1},$$

where, in line (10), we have applied our Power Rule to x^{-n}. This is permissible since $-n$ is a positive integer and our current power rule would then apply.

Thus, we have shown that if $n \in \mathbb{Z}$ is a negative integer, then

$$\frac{d\,x^n}{dx} = nx^{n-1} \tag{11}$$

This formula has *exactly* the same form as our (old) Power Rule.

At this point, it is traditional to do a little hand waving. The power rule, which is valid for integer exponents is, in fact, valid for *all fractional exponents* as well. I'll go ahead and raise the banner on this form of the power rule so we (read you) can begin practicing.

Extended The Power Rule: JG.
Let $r \in \mathbb{Q}$, the set of all rational numbers, then

$$\frac{d\,x^r}{dx} = rx^{r-1}.$$

It is this formula that will be our working formula. The Power Rule needs to be memorized. You must train yourself to recognize power functions, and you must have the ability to utilize the formula *correctly* — most important!

Here are some quick visual examples.

$$\frac{d\,x^{-3}}{dx} = -3x^{-4} \qquad \frac{d\,x^{1/2}}{dx} = \frac{1}{2}x^{-1/2}$$

$$\frac{d\,s^{-1/2}}{ds} = -\frac{1}{2}s^{-3/2} \qquad \frac{d\,w^{7/3}}{dw} = \frac{7}{3}w^{4/3}$$

Visually scan these examples. Do you see the pattern of solution in each? Verbalize each example: the derivative of x^{-3} is the exponent -3 times x raised to one less power x^{-4}. Thus we get

$$\frac{d\,x^{-3}}{dx} = -3x^{-4}.$$

Example 7.7. (Skill Level 1) Calculate $\dfrac{d}{dx}\dfrac{5}{x^4}$.

Example 7.8. (Skill Level 1) Calculate $\dfrac{d}{dx}x\sqrt{x}$.

Exercise 7.5. Calculate $\dfrac{d}{dx}\dfrac{6}{x^2\sqrt{x}}$.

Exercise 7.6. (Skill Level 1) Calculate $\dfrac{d}{dx}\dfrac{x^2+1}{\sqrt{x}}$.

Example 7.9. (Skill Level 1) Calculate $\dfrac{d}{dx}\dfrac{x^2+1}{\sqrt{x}+1}$.

7.7. The Differential Notation

Let $y = f(x)$ be a differentiable function. We have seen that the notation

$$\frac{dy}{dx} = f'(x) \tag{12}$$

is the Leibniz notation for the derivative of the function f with respect to the variable x. Up to this point the notation was taken as a whole — a single unit; however, the notation itself suggests some sort of *division process*. In this section we shall explore this observation.

We want to take (12), uncouple the numerator from the denominator, enabling us to think of (12) as a ratio rather than a single unit of notation. To do this, we do the obvious: We make a definition!

Let $y = f(x)$ be a differentiable function of x. Then x is the *independent variable* and y is the *dependent variable*. Define a *new independent variable* to be represented by the symbol dx. The variable dx is called the *differential* of x. Now we want to define a *dependent variable* called dy. The variable dy will be dependent since its value will depend on the value of dx. The value we choose for dy is exactly the one that will make the ratio of dy by dx equal to $f'(x)$. More explicitly, for a given value of x and a given value of dx, we want to the value of dy to be

$$dy = f'(x)\,dx. \tag{13}$$

If the value of dy is calculated by the equation in (13), then, for $dx \neq 0$,

$$\frac{dy}{dx} = f'(x).$$

That is, if dy has the value, $dy = f'(x)\,dx$, then, naturally, the ratio of dy by dx is exactly $f'(x)$ — as desired. Let us formalize these ruminations into a definition.

Definition 7.6. Let $y = f(x)$ be a differential function of x. Define dx to be an independent variable, and define dy to be a dependent variable such that

$$dy = f'(x)\,dx. \tag{14}$$

The variable dx is called the *differential* of x, and dy is called the *differential* of y. Alternately, we also write,

$$df = f'(x)\, dx.$$

In this case, we say that df is the *differential* of the function f.

Definition Notes: Of course, dy and df are the same quantity. The dy version is used most often for named functions as well as anonymous functions.

▪ The differential, given in (14), has several interpretations and applications. We will eventually explore them all. At this point, we present the differential as a uncoupled version of the Leibniz derivative notation.

▪ As usual, the choice of the letters to designate the independent and dependent variables is unimportant; what is important is the concept of the relationship between the differential of the independent variable and the differential of the dependent variable. For example, suppose $s = g(w)$; this statement asserts that s is considered the dependent variable, and w the independent variable. Therefore, (14) defines

$$ds = g'(w)\,dw.$$

Additional comments will be made in this regard later. ∎

Calculation. Now we explore the simple act of calculation. Here are a number of quick visual examples:

1. $y = x^3 \implies dy = 3x^2\,dx.$
2. $w = s^5 \implies dw = 5s^4\,ds.$
3. $A = \pi r^2 \implies dA = 2\pi r\,dr.$
4. $q = \sin(v) \implies dq = \cos(v)\,dv.$
5. $f(x) = x^{3/2} \implies df = \dfrac{3}{2}x^{1/2}\,dx.$

As you can see, the act of calculating a differential is the height of triviality—given that you can differentiate the functional relationship. In all cases, the new "differential variables" are set up so that their ratio is always the derivative of the function. Take as an example, (1) above:

$$y = x^3 \implies dy = 3x^2\,dx \implies \frac{dy}{dx} = 3x^2,$$

the latter equation is representative of the derivative of $y = x^3$, as we have seen.

By the way, can use the differential notation to display and calculate derivatives as well. We can write,

$$d\,x^3 = 3x^2\,dx.$$

This equation makes the statement that the derivative of the function x^3 is $3x^2$, because, when we divide both sides of the equation by dx we get

$$\frac{d\,x^3}{dx} = 3x^2.$$

Similarly, the statement

$$d\,\sin(x) = \cos(x)\,dx$$

should be interpreted as *equivalent to* the formula for the derivative of the sine function:

$$d\,\sin(x) = \cos(x)\,dx \iff \frac{d}{dx}\sin(x) = \cos(x).$$

EXERCISE 7.7. Let $y = x^7$, $w = \sqrt{s}$, $V = \dfrac{4}{3}\pi r^3$. Calculate dy, dw, and dV.

Algebraic Properties of the differential. Because the differential is just the derivative, with the "dx" cleared out of the denominator, one would expect that the differential has properties similar to the properties of derivative — and you would be right.

Let u and v be differentiable functions of some (unspecified) independent variable.

1. (Homogeneity Property) For any constant $c \in \mathbb{R}$,
$$d(cu) = c du.$$
2. (Additive Rule)
$$d(u + v) = du + dv.$$
3. (Produce Rule)
$$d(uv) = u\, dv + v\, du.$$

4. (Quotient Rule)
$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

These are identical to their derivative counterparts; however, they have the advantage that they are more compact and it isn't necessary to explicitly specify the independent variable. This *differential notation* is used quite extensively in *differential equations*, a course, some day, you may take.

EXAMPLE 7.10. Calculate $d(x^2 \sin(x))$.

EXERCISE 7.8. Calculate $d(x\sqrt{x})$.

EXERCISE 7.9. Calculate $d\left(\dfrac{x^2}{x^2+1}\right)$.

The language of differentials can be used to pose questions that are meant to aggravate the minds of freshmen.

Exercise 7.10. I'm thinking of a function y of x having the property $dy = x^4\,dx$. What function am I thinking of ... I'll bet you'll never, never guess!

Simple Interpretation. Rise over Run.
When the independent variable is not given a value, the differential serves very well as an alternate notation to the usual differentiation notation. Thus,

$$\frac{dy}{dx} = f'(x) \text{ is equivalent to } dy = f'(x)\,dx,$$

but, just as the derivative can be evaluated numerically, so can the differential. The interpretation of the differential comes, naturally, from that of the derivative.

Take a concrete example to illustrate the concept. Let

$$y = f(x) = x^2 \qquad f'(x) = \frac{dy}{dx} = 2x \qquad dy = 2x\,dx.$$

Now, evaluate each of these at $x = 3$:

$$x = 3 \implies y = f(2) = 9 \qquad f'(3) = \frac{dy}{dx} = 6 \qquad dy = 6\,dx.$$

Now $f'(3) = 6$ is interpreted as the slope of the line tangent to the graph of f at the point corresponding to $x = 3$: that's the point $(\,3, 9\,)$. So the slope of the tangent line is $m = 6$.

Recall. Let's unarchive our straight line knowledge. Recall, that one interpretation of the slope, m, of any line is that

$$m = \frac{\text{rise}}{\text{run}}. \tag{15}$$

Slope is calculated as rise over run. That is, if we start at a point on the line, and move horizontally ("run") a certain amount, to get back to the line, we must move vertically ("rise") a certain amount. Whatever we "run," we must "rise" in such a way that (15). Given the amount of "run," how much do we "rise"? The answer is,

$$\text{run} = m(\text{rise}). \tag{16}$$

Rather than using the silly notation of "rise" and "run," let's use different symbolisms. When we move horizontally, we are moving parallel to the x-axis, let the amount of "run" be designated by dx. When we "rise" we are moving vertically, hence parallel to the y-axis, so let me designate "rise" by the symbol dy. Thus,

$$dx = \text{rise}$$
$$dy = \text{run}$$

Replacing these new symbols into equation (16) we get

$$dy = m\,dx. \tag{17}$$

End of Recall.

Now, we put these recollections and new notation back into context of our discussion. The function $f(x) = x^2$ had $f'(x) = 2x$, and $f'(3) = 6$. The slope of the tangent line is $m = 6$. For this line, (17) becomes,

$$dy = 6\,dx.$$

This equation is (1) the differential of the function at $x = 3$, (2) tells us that if we move dx in the x-direction, then to get back to the (tangent) line we must move dy in the y-direction. In particular, if we put, $dx = 2$, that is, if we start at any point on the tangent line, and move $dx = 2$ units to the *right*, then we must move $dy = 6(2) = 12$ units vertically upwards to get back to the line. Similarly, a value of $dx = -3$ means that we are moving 3 units horizontally to the *left*, in this case, $dy = 6(-3) = -18$; this means we must move 18 units vertically downward to get to the line.

Final Interpretation. Having seen the interpretation of dy and dx, let's formalize this in abstract. Given,

$$dy = f'(x)\,dx,$$

then on the tangent line corresponding to the value of x, if we "run" away from the tangent line by an amount of dx (positive or negative), then dy is the amount of "rise" we must make to get back to the tangent line.

Exercise 7.11. Consider $y = \sqrt{x}$.

a. Find the differential of y.
b. Find the differential of y corresponding to $x = 4$.
c. How much does the tangent line rise if we were to move from $x = 4$ to $x = 4.1$?
d. How much does the tangent line rise if were to move from $x = 4$ to $x = 3.8$.

There is another easy interpretation of $dy = f'(x)\,dx$, when x has been given a numerical value, say, $x = a$: $dy = f'(a)\,dx$. This interpretation is in regards to the equation of the tangent line.

EXAMPLE 7.11. Let $y = x^2$. Then as we have seen, $dy = 2x\,dx$. For $x = 3$, this differential becomes $dy = 6\,dx$. Can you interpret $dy = 6\,dx$ as the equation of the line tangent to $y = x^2$ at $x = 3$?

EXERCISE 7.12. Find the equation of the line tangent to the graph of $y = x^3$ at the point on the graph corresponding to $x = 1$. Use the techniques of EXAMPLE 7.11.

Section Summary. Let $y = f(x)$ be a differentiable function.

■ We have defined the *differential of f* as

$$dy = f'(x)\,dx$$

On the face of this, this differential notation is an alternate notation to the derivative notation.

■ The utility of the differential, is that enables us to think of the Leibniz notation, dy/dx, as a ratio: the differential of y divided by the differential of x. This allows us to manipulate derivative concepts algebraically.

■ The differential satisfies the same rules of combining functions as the derivative does. See Properties. These properties can be exploited to calculate differential in same way as derivatives.

■ The simplest interpretation of

$$dy = f'(x)\,dx$$

is the “rise over the run” interpretation. If we are on some tangent line to the graph of f, and we move an amount of dx parallel to the

x-axis, then we must move dy parallel to the y-axis to get back to the tangent line.

■ Another interpretation of

$$dy = f'(x)\,dx$$

is that it represents the equation of the tangent line, where the xy-axis system has been translated to the point of tangency, and the name of the new axis system is the $dx\,dy$-axis system. See Example 7.11 for details.

7.8. Practicing the Mechanics

In this section we practice the mechanics of differentiation. But first, I would offer some advice.

How to Differentiate.

■ When you look at the function to be differentiated, you must first ask yourself a series of questions. Is this a power function? (If yes, apply the Power Rule.) Is this function a sum or difference of two functions? (If yes, apply the Additive Rule.) Is this function a product of two functions? (If yes, apply the Product Rule.) Is this function a quotient of two functions? (If yes, apply the Quotient Rule.) No matter how complicated the functions become, at each stage of the differentiation process, this classification can be made. Make it.

▪ *Before differentiating*, always look at the function to be operated on. Possibly do some algebraic manipulation for the purpose of simplification or for the purpose of preparing the function for differentiation. For example, the function $f(x) = x\sqrt{x}$ is indeed a product of two functions, as per item (a) above, but it would be silly to apply the product rule. The reason for this is the function f can be rewritten as

$$f(x) = x\sqrt{x} = xx^{1/2} = x^{3/2}.$$

This function can be more easily and efficiently differentiated when treated as a power function.

▪ As you apply the rules, *verbalize* them. This will help you remember them. These formulas *must* be memorized! (Sorry, $\mathfrak{DP}$)

▪ Until you become a *master of differentiation*, do not apply more than one rule per step. (You are a *master* when you do *not* have to look in the back of the book for the answer.)

- Apply the rules carefully and methodically. Use correct notation — very important.
- Always practice your algebra. Simplify your answer.
- Accumulate a history of *problem solving*. The more problems you solve, the greater your history of problem solving. Having a history of problem solving enables you to tackle new problems with more confidence and authority.
- See patterns in the sand. Many problems are just repetitions of the same basic problem. Do not treat each problem as a unique problem you have never seen before. Create memory pointers: try to associate problem types so you can use your past experience to guide you on each new quest. For example (trivial): If you first differentiate $y = x^2$ to get $y' = 2x$, and then you differentiate $y = x^3$ to get $y' = 3x^2$, and then you differentiate $y = x^4$ to get $y' = 4x^3$, and then you are asked to differentiate $y = x^5$ wouldn't you say, "I have already done this problem!" All the above mentioned problems are, in fact, the same problem over and over again; they are all just power functions, they are all solve using the power rule. Over and over again.

How to do Homework.

▪ When you work on problems at home, use plenty of paper, give yourself plenty of room to do each problem. Do not squeeze your work into a small space (or a large space). Write *horizontally* across the sheet of paper. Write neatly. *Get control of your hand.*

▪ When doing your homework, do not be sloppy or lazy. Solve a problem as though your *job* depended on it. Use correct notation. I have seen many students homework, quite generally, it is unorganized, incomplete, and not written very well. Every point in part (a) is violated. Sometimes the attitude of the student is "I'll do it this way for my personal use, but on the test, I'll be neat and do things correctly." I'm sorry, it doesn't work that way. On a test, you will solve problems exactly the same way you solved problems at home — only there is no back of the book to provide answers to guide your thinking and steps; and there is no solution manuals to use; and, o.k., there are no *hyper-links* to the solutions either.

▪ Do all problems as if *I* am looking over your shoulder, and you want to impress me.

■ I used to solve problems in a scrap sheet of paper, then transfer them neatly into a notebook. This gave me the opportunity to rewrite the problem neatly and in an organized way, but it also allowed me to rethink my solution, and to recheck my calculations.

■ How do we solve problems? One way is by example. Where do I get examples, you ask? Your textbook provides examples, read them. Quite often, examples in each section of a textbook illustrate how to solve different problem types that appear in the exercise section. Any notes obtained during class sessions provide a valuable resource of examples. Reread your notes; be aware of the examples done in class. These notes represent a large number of examples; many solutions exhaustively worked out for you.

■ But wait. The point of the previous paragraph needs to be extended. Not only do these examples provide solutions to problem types, but also provide illustrations of *good style*, *correct notation*, *fine organization*, and *well thought out presentation*. These features are important to your instructor, and to you, it is hoped.

Other Advice.

- Take pride in your work. Take the attitude that each problem has something to teach you. Adopt the personal philosophy: "I want to do all things correctly and with good technique." "I want to understand *why* things are as they are."
- Study and work for the purpose of *understanding* the ideas and *mastering the techniques*, rather than to learn enough to get by. Our country needs people who are competent. Teaching yourself to solve problems yourself, studying for comprehension all go toward this goal.
- Work hard. Put in 100%. This is necessary in order to find out what you are intellectually capable of doing. Isn't that what you should be working towards?

Enough of such advice. Let's bring on the examples!

⇓ Click here to continue.

The Power Rule JG.
Let $n \in \mathbb{N}$, i.e. $n = 1, 2, 3, \ldots$, then

$$\frac{d\,x^n}{dx} = nx^{n-1}.$$

The derivative of x raised to a power, is that power times x raised to one less power.

Homogeneity of Differentiability.

$$\frac{d(cu)}{dx} = c\frac{du}{dx}$$

The derivative of a constant times a function is that constant times the derivative of the function.

The Additive Property of Derivative

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

The derivative of the sum of two functions is the sum of the derivatives.

The Produce Rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of a product is the first times the derivative of the second, plus the second times the derivative of the first.

The Quotient Rule

$$\frac{d}{dx}\frac{u}{v} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad v \neq 0.$$

The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.

The Extended Power Rule

Let $r \in \mathbb{Q}$, the set of rational numbers, then

$$\frac{d\,x^r}{dx} = rx^{r-1}.$$

The derivative of x raised to a power, is that power times x raised to one less power.

The Chain Rule Let $y = f(u)$ and $u = g(x)$ be differentiable and compatible for composition, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The derivative of a composition of two functions is the derivative of the outer function times the derivative of the inner function.

Generalized Power Rule

Let u be a function of x and $r \in \mathbb{Q}$, the set of rational numbers, then

$$\frac{d\,u^r}{dx} = ru^{r-1}\frac{du}{dx}.$$

The derivative of a function raised to a power, is the power times the base function raised to one less power, times the derivative of the base function.

Solutions to Exercises

3.1. I'll follow my own advice, except I'll cut and paste the solutions, and make all appropriate changes.

Solution to (a): Calculate the difference quotient. We proceed as follows.

1. Gather the Needed Information.

Difference Quotient: $\dfrac{f(a+h) - f(a)}{h}$.

The Function and point: $f(x) = 5x^2$-2, and $a = 1$.

2. Make Calculations: Build up the difference quotient.

$$f(a) = f(1) = 3$$
$$f(a+h) = f(1+h) = 5(1+h)^2 - 2$$

and so,

$$\begin{aligned} f(a+h) - f(a) &= (5(1+h)^2 - 2) - 3 \\ &= 5(1 + 2h + h^2) - 5 \\ &= 5 + 10h + 5h^2 - 5 \\ &= 10h + 5h^2. \end{aligned}$$

Where we have been true to our algebraic roots and simplified the above expressions in an orderly fashion.

And finally, the difference quotient

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{10h + 5h^2}{h} \\ &= 10 + 5h. \end{aligned}$$

Thus,

$$\boxed{\frac{f(a+h) - f(a)}{h} = 10 + 5h,} \tag{A-1}$$

we have constructed in a well-organized way, the *difference quotient*.

Note: I was not satisfied with just "plugging" the function into the formula (3) and getting the hideous expression

$$\frac{5(1+h)^2 - 2 - 3}{h}.$$

This is the difference quotient but it does not advance the ultimate problem forward (Part (b)). We would not have been true to our *algebraic roots*!

Solution to (b): All the heavy lifting has been down in the separate step of calculating *and simplifying* the difference quotient.

Gather Needed Information.

The definition: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

The Function and Point: $f(x) = 5x^2 - 2$ and $a = 1$.

Make the Calculations.

$$\begin{aligned} f'(1) = f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} 10 + 5h \qquad \triangleleft \text{ Part (a), equation (A-1)} \\ &= 10 \end{aligned}$$

Thus,

$$\boxed{f'(1) = 10.}$$

Organized work yields an organized mind.

Solution to (c) We just "plug" our information into the general formula developed in (5). That general formula is

$$y = f(a) + f'(a)(x - a).$$

This is the equation of the line tangent to the graph of f at $x = a$.

Our function is $f(x) = 5x^2 - 2$, our point of interest is $x = 1$ (this is our a, i.e. $a = 1$).

Auxiliary Calculations:

$$a = 1$$
$$f(1) = 5(1)^2 - 2 = 3$$
$$f'(1) = 10 \qquad \text{from part (b)}$$

The Equation of Tangent Line: Now we have all the "input data" we need to utilize (5): Taking and inserting this information into (5) we obtain

$$y = 3 + 10(x - 1).$$

or

$$\boxed{y = 10x - 7.}$$

This is the equation of the line tangent to the graph of f at $x = 1$.

Exercise Notes: I told you it was a cut and paste job. The point was that the steps or procedures are the same. We use exactly the same train of thought as in the previous examples. By seeing a consistency, or a unified method, of solution it is hoped you will enter these problems with confidence, authority, and a visualization of the sequence of steps needed to solve the problem.

▪ Of course, not all tangent lines pass through the point $P(1, 3)$; I hope you don't jump to that conclusion. I chose the function in Example 3.1 and the function in this exercise just to confuse you. I hope you were not! 𝔇𝔓 ▪

Exercise 3.1. ▪

3.2. Here is an outline of details. Recall $f(x) = 5x^2 - 2$.

Difference Quotient:

$$\frac{f(x+h) - f(x)}{h} = \frac{(5(x+h)^2 - 2) - (5x^2 - 2)}{h} = 10x + 5h^2.$$

The Derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} 10x + 5h^2 \\ &= 10x \end{aligned}$$

Thus,

$$\boxed{f(x) = 5x^2 - 2 \qquad f'(x) = 10x.}$$

Exercise Notes: Are these results consistent with the results of Exercise 3.1? ■

Exercise 3.2. ■

3.3. Here is an outline of the details.

Difference Quotient:

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{3(x+h)}-\sqrt{3x}}{h} \\
&= \frac{\sqrt{3(x+h)}-\sqrt{3x}}{h}\,\frac{\sqrt{3(x+h)}+\sqrt{3x}}{\sqrt{3(x+h)}+\sqrt{3x}} \\
&= \frac{3}{\sqrt{3(x+h)}+\sqrt{3x}}
\end{aligned}$$

The Derivative:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)}+\sqrt{3x}} \\
&= \frac{3}{2\sqrt{3x}}
\end{aligned}$$

Thus,

$$\boxed{f'(x) = \frac{3}{2\sqrt{3x}}.}$$

Domain Analysis: $\text{Dom}(f') = (\,0, +\infty\,)$. Exercise 3.3. ■

3.4. For $f(x) = \dfrac{1}{\sqrt{x}}$, we proceed along standard lines of inquiry.

Difference Quotient:

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{1}{h}(f(x+h)-f(x)) \\
&= \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) \\
&= \frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right) \\
&= \frac{1}{h}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) \\
&= \frac{1}{h}\left(\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\right) \\
&= \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \qquad \text{(A-2)}
\end{aligned}
$$

That was a bit of a bear!

The Derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} && \text{from (A-2)} \\ &= \frac{-1}{x(2\sqrt{x})} \\ &= \frac{-1}{2x\sqrt{x}} \end{aligned}$$

Thus,

$$\boxed{f'(x) = \frac{1}{2x\sqrt{x}}.}$$

Exercise Notes: The only difference between this exercise and Example 3.9 or Exercise 3.3 is the volume of algebra needed for this problem. The step were the same, the conjugate trick was used in all problem. The difference is the *algebra*; therefore, to succeed in *Calculus* you must constantly and consistently be improving your *algebra*!

■

Exercise 3.4. ■

3.5. We calculate the left- and right-hand derivatives of

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x & x > 0 \end{cases}$$

Left-hand Derivative:

$$\begin{aligned} f'_{-}(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{h^2}{h} \\ &= \lim_{h \to 0^-} h \\ &= 0 \end{aligned}$$

Right-hand Derivative:

$$
\begin{aligned}
f'_+(0) &= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \\
&= \lim_{h\to 0^+} 1 \\
&= 1
\end{aligned}
$$

Thus,

$$f'_-(0) = 0 \neq 1 = f'_+(0).$$

This means (1) that f is not differentiable at $x = 0$ (see *Theorem Notes* following Theorem 3.5); and (2) that f has a corner at $x = 0$.

Finally note that the function f is continuous at $x = 0$ because

$$f(0) = 0 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x).$$

Exercise 3.5. ■

3.6. Begin by calculating the left-hand derivative at $x = 0$.

Left-Hand Derivative:

$$
\begin{aligned}
f'_-(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0^-} \frac{h^{1/3}}{h} \\
&= \lim_{h \to 0^-} \frac{1}{h^{2/3}} \\
&= +\infty.
\end{aligned}
$$

The limit is $+\infty$ since the h is squared: $h^{2/3} = (h^{1/3})^2$; consequently, we are taking the limit of a positive quantity, where the denominator goes to zero.

Right-hand Derivative:

$$\begin{aligned} f'_+(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^+} \frac{h^{1/3}}{h} \\ &= \lim_{h \to 0^+} \frac{1}{h^{2/3}} \\ &= +\infty. \end{aligned}$$

As you can see, the $f'_+(0)$ calculation was the same as the $f'_-(0)$ calculation.

Thus, we have shown that

$$f'_-(0) = f'_+(0) = +\infty,$$

this is my definition of a vertical tangent.

Finally, recall that $f(x) = x^{1/3}$ is an algebraic function is continuous.

Exercise 3.6. ■

4.1. The name of the function is f', the name of the derivative function is the same as the parent function f' with a $'$ post-fixed. This becomes f''. Got it?

Exercise 4.1. ■

6.1. The answer is

$$\left.\frac{d\,5}{dx}\right|_{x=123.45} = 0.$$

Because the right-hand side of $\dfrac{d\,5}{dx} = 0$ does not have an 'x' in it, the right-hand side, which is a function of x, must be a constant function (constantly equal to 0). Therefore, the evaluation of the function $\dfrac{d\,5}{dx}$ at a particular value of x, will yield 0.

Exercise 6.1. ■

7.1. Does your neatly written out solution look like this? Or have you just scribbled down the answer?

$$\begin{aligned}
\frac{d}{ds}4s^5 - \frac{2}{3}s^3 &= \frac{d}{ds}4s^5 - \frac{d}{ds}\frac{2}{3}s^3 \\
&= 4\frac{d\,s^5}{ds} - \frac{2}{3}\frac{d\,s^3}{ds} \\
&= 4(5s^4) - \frac{2}{3}(3s^2) \\
&= 20s^4 - 2s^2 \\
&= 2s^2(10s^2 - 1)
\end{aligned}$$

Exercise 7.1. ■

7.2. Let's look at each part. From the solution to EXERCISE 7.1, we have

$$w = 4s^5 - (2/3)s^3 \qquad \frac{dw}{ds} = 2s^2(10s^2 - 1).$$

Part (a): We want to find were the tangent line is horizontal. A horizontal line has slope of 0; therefore, we want to find were the slope of the tangent line is 0. The first derivative of a function is the slope of the tangent line; therefore, we want to find where the first derivative of the function is equal to 0. We ask this question by setting up the equation:

$$\begin{aligned} \frac{dw}{ds} &= 0 && \text{solve for } s \\ 2s^2(10s^2 - 1) &= 0 && \text{solve for } s \end{aligned}$$

Now if you haven't completely solved this problem, now that I have properly formulated it, *you* finish it off.

Which of the following is the answer to (a)?

(a) $s = 2,\ 0$ (b) $s = 0$ (c) $s = \pm\dfrac{1}{10}$ (d) $s = 0,\ \pm\dfrac{1}{10}$

Part (b): Now we want to find all s at which the tangent line is parallel to the line $w = 120s + 3$. Our function, $w = 4s^5 - (2/3)s^3$, defines w as a function of s; therefore we consider s the independent variable and w the dependent variable; consequently, for

$$\frac{d}{ds} 120s + 3 = 120.$$

So, the line has a constant slope of $m = 120$. We want to find all points s at which the slope of the tangent line is 120. We set up the defining equation

$$\frac{dw}{ds} = 120 \qquad \text{solve for } s$$

$$2s^2(10s^2 - 1) = 120 \qquad \text{solve for } s \tag{A-3}$$

Now if you have not already properly formulated the problem. All you have to do now is to solve equation (A-3) for s — mere algebra. (I'm sure you are a *master of algebra*!)

Which of the following as the answers to (b)? (Work the problem out first before daring to peek!)

(a) $s = \pm\sqrt{5/2}$ (b) $s = \dfrac{5}{2}$ (c) $s = 1, -3$ (d) $s = \pm\sqrt{12/5}$

Don't give up on this problem. Keep trying to solve equation (A-3). You never know, the problem may teach you something that later you can use. Obtain a *history of problem solving* so that you will have the experience and confidence to tackle problems encountered in the future.

Exercise 7.2. ■

7.3. For $y = x^{12}(1+x^2)$, we want to differentiate the product of two functions. Use the Product Rule, that's what it is for.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}x^{12}(1+x^2)\\ &= x^{12}\frac{d}{dx}1+x^2+(1+x^2)\frac{d\,x^{12}}{dx}\\ &= x^{12}(2x)+(1+x^2)12x^{11}\\ &= 2x^{11}(x^2+6(1+x^2))\\ &= 2x^{11}(7x^2+6)\end{aligned}$$

Thus,

$$\boxed{\frac{dy}{dx} = 2x^{11}(7x^2+6).}$$

Example Notes: Again, notice the use of the $\dfrac{d}{dx}$ notation to work through this problem.

▪ This problem could have been solve more easily as follows:

$$y = x^{12}(1 + x^2) = x^{12} + x^{14}$$
$$y' = 12x^{11} + 14x^{13} = 2x^{11}(6 + 7x^2).$$

Normally, this would have been the method of solution for this problem. I required the use of the product rule so you can start getting practice.

▪ Later, the product rule will become an essential tool for differentiation. We have to wait until we can build up our set of rules. ▪

Exercise 7.3. ▪

7.4. This is a quotient, so use the Quotient Rule.

$$\begin{aligned}
\frac{d}{dx}\frac{7x^6}{6x^8-1} &= 7\frac{d}{dx}\frac{x^6}{6x^8-1} && \triangleleft \text{ Homogen.}\\
&= 7\frac{(6x^8-1)\frac{d\,x^6}{dx} - x^6\frac{d}{dx}(6x^8-1)}{(6x^8-1)^2} && \triangleleft \text{ Quot.}\\
&= 7\frac{(6x^8-1)(6x^5) - x^6(48x^7)}{(6x^8-1)^2}\\
&= 7\frac{6x^5(6x^8-1-8x^8)}{(6x^8-1)^2}\\
&= -\frac{42x^5(2x^8+1)}{(6x^8-1)^2}.
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}\frac{7x^6}{6x^8-1} = -\frac{42x^5(2x^8+1)}{(6x^8-1)^2}.}$$

Exercise 7.4. ■

7.5. Let's hope you learned your lessons from the previous examples. Note that

$$\frac{6}{x^2\sqrt{x}} = 6x^{-5/2}.$$

Thus,

$$\begin{aligned}\frac{d}{dx}\frac{6}{x^2\sqrt{x}} &= \frac{d}{dx}(6x^{-5/2}) \\ &= (6)(-\frac{5}{2})x^{-7/2} \\ &= -15x^{-7/2} \\ &= -\frac{15}{x^{7/2}} \\ &= -\frac{15}{x^3\sqrt{x}}.\end{aligned}$$

Finally,

$$\boxed{\frac{d}{dx}\frac{6}{x^2\sqrt{x}} = -\frac{15}{x^3\sqrt{x}}.}$$

I hope you didn't use the quotient rule.

Exercise 7.5. ■

7.6. Again, we can avoid the quotient rule.

$$\begin{aligned}\frac{x^2+1}{\sqrt{x}} &= \frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}} \\ &= \frac{x^2}{x^{1/2}} + \frac{1}{x^{1/2}} \\ &= x^{3/2} + x^{-1/2}\end{aligned}$$

If you are an *algebraic wizard*, the above steps would be instantaneous.

Now we can differentiate

$$\begin{aligned}\frac{d}{dx}\frac{x^2+1}{\sqrt{x}} &= \frac{d}{dx}x^{3/2} + x^{-1/2} \\ &= \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2} \\ &= \frac{1}{2}x^{-3/2}(3x^2-1).\end{aligned}$$

Make sure you understand the last algebraic step, I factored out the quantity $(1/2)x^{-3/2}$. It is important that you *constantly* improve your algebra.

$$\boxed{\frac{d}{dx}\frac{x^2+1}{\sqrt{x}} = \frac{1}{2}x^{-3/2}(3x^2-1).}$$

Exercise 7.6. ■

7.7. Based on the definitions of each function, we automatically, classify each variable as *independent* or *dependent*.

$$dy = 7x^6\,dx \qquad dw = \frac{1}{2\sqrt{x}}\,ds \qquad dV = 4\pi r^2\,dr.$$

Exercise 7.7. ■

7.8. I hope you didn't use the product rule!

$$d(x\sqrt{x}) = dx^{3/2} = \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}\sqrt{x}\,dx,$$

use the power rule!

Exercise 7.8. ■

7.9. This is the differential of a quotient.

$$d\left(\left(\frac{x^2}{x^2+1}\right) = \frac{(x^2+1)\,dx^2 - x^2\,d(x^2+1)}{(x^2+1)^2}\right.$$

$$= \frac{(x^2+1)(2x\,dx) - x^2(2x\,dx)}{(x^2+1)^2}$$

$$= \frac{2x(x^2+1) - 2x^3}{(x^2+1)^2}\,dx$$

$$= \frac{2x}{(x^2+1)^2}\,dx$$

These are virtually the same details you would have generated had the problem been to calculate

$$\frac{d}{dx}\frac{x^2}{x^2+1}$$

and the answer would have been the same, after dividing through by dx.

Exercise 7.9. ■

7.10. Hummm! A function whose differential looks like $dy = x^4\,dx$. This means

$$\frac{dy}{dx} = x^4.$$

What function gives a derivative of x^4. After many hours of meditation hopefully, you'll arrive at something like

$$y = \frac{1}{5}x^5.$$

You would be right, but this is not the function that I was thinking of!

I was thinking of the function

$$y = \frac{1}{5}x^5 + 17.$$

Are we both right? Say YES!

Differential equations concerns itself with identifying (unknown) functions the only clues to whose identity is that the function satisfy some *differential equation*: An equation involving differentials or derivatives, like this one

$$dy = x^5\, dx.$$

O.k., this was an easy one. There are harder ones too. Some physical systems can be described in terms of these differential equations. If we can solve the differential equation we have characterized the system.

Exercise 7.10. ■

7.11. This is a straight-forward application of the current discussion.

Solution to (a): $dy = \dfrac{1}{2\sqrt{x}}\,dx$. This is because

$$\frac{dy}{dx} = \frac{d\,x^{1/2}}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}},$$

by the Power Rule.

Solution to (b): Put $x = 4$,

$$dy = \left.\frac{1}{2\sqrt{x}}\,dx\right|_{x=4} = \frac{1}{4}\,dx$$

Solution to (c): On the tangent line to the graph of $y = x^2$ at $x = 4$, move horizontally from $x = 4$ to $x = 4.1$. This means we “run” by an amount of $dx = .1$. Then

$$dy = \left.\frac{1}{4}\,dx\right|_{dx=.1} = \frac{1}{4}(.1) = \frac{1}{40}.$$

Thus, if we move $dx = .1$ to the right, we must move up $dy = 1/40$ (upward) to get to the tangent line.

Solution to (d): On the tangent line to the graph of $y = x^2$ at $x = 4$, move horizontally from $x = 4$ to $x = 3.8$. This means we "run" by an amount of $dx = 3.8 - 4 = -.2$, to the *left* Then,

$$dy = \frac{1}{4}\,dx\Big|_{dx=-.2} = \frac{1}{4}(-.2) = -\frac{1}{20}.$$

Thus, if we move $dx = -.2$ to the left, we must move up $dy = -1/20$ (downward) to get to the tangent line.

Exercise 7.11. ■

7.12. First make all routine calculations:

$$y = x^3$$
$$dy = 3x^2\,dx$$

At $x = 1$,

$$y = 1$$
$$dy = 3\,dx \tag{A-4}$$

The point of tangency is $(\,x_0, y_0\,) = (\,1, 1\,)$. Transformation equations are

$$\begin{aligned} dx &= x - x_0 = x - 1 \\ dy &= y - y_0 = y - 1. \end{aligned} \tag{A-5}$$

Substituting (A-5) back into (A-4) we get

$$\boxed{y - 1 = 3(x - 1).}$$

This is the equation of the line tangent to the graph of $y = x^3$, at the point on the graph corresponding to $x = 1$. Exercise 7.12. ■

Solutions to Examples

3.1. This is where we can apply our mechanical skills concerning functions.

Solution to (a): Calculate the difference quotient. We proceed as follows.

1. Gather the Needed Information.

Difference Quotient: $\dfrac{f(a+h)-f(a)}{h}$.

The Function and point: $f(x) = 3x^2$, and $a = 1$.

2. Make Calculations: Build up the difference quotient.

$$f(a) = f(1) = 3$$
$$f(a+h) = f(1+h) = 3(1+h)^2$$

and so,

$$\begin{aligned} f(a+h) - f(a) &= 3(1+h)^2 - 3 \\ &= 3(1 + 2h + h^2) - 3 \\ &= 3 + 6h + 3h^2 - 3 \\ &= 6h + 3h^2. \end{aligned}$$

Where we have been true to our algebraic roots and simplified the above expressions in an orderly fashion.

And finally, the difference quotient

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{6h + 3h^2}{h} \\ &= 6 + 3h. \end{aligned}$$

Thus,

$$\boxed{\frac{f(a+h) - f(a)}{h} = 6 + 3h,} \tag{S-1}$$

we have constructed in a well-organized way, the *difference quotient.*

Note: I was not satisfied with just "plugging" the function into the formula (3) and getting the hideous expression

$$\frac{(1+h)^2-3}{h}.$$

This is the difference quotient but it does not advance the ultimate problem forward (Part (b)). We would not have been true to our *algebraic roots*!

Solution to (b): All the heavy lifting has been down in the separate step of calculating *and simplifying* the difference quotient.

Gather Needed Information.

The definition: $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$.

The Function and Point: $f(x) = 3x^2$ and $a = 1$.

Make the Calculations.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} 6 + 3h \qquad \triangleleft \text{ Part (a), equation (S-1)} \\ &= 6 \end{aligned}$$

Thus,

$$\boxed{f'(1) = 6.}$$

Organized work yields an organized mind.

Example 3.1. ■

3.2. We must construct the equation of a straight line. To do this, we must unarchive our straight line knowledge. Recall, the point-slope form of the equation of a line:

$$y - y_0 = m(x - x_0). \tag{S-2}$$

This is the equation of the line of slope m passing through the point $P(\,x_0, y_0\,)$.

Therefore, to construct our line we need a point and a slope.

The Point: We want our line to be tangent to the graph of f at the *point* corresponding to $x = a$? What does that mean? We want our line to pass through the point $(\,a, f(a)\,)$.

$$\text{The Point: } P(\,a, f(a)\,).$$

The Slope: We want our line to be *tangent* to the graph of f at the point corresponding to $x = a$. The slope of the tangent line of f at $x = a$ is our interpretation of $f'(a)$.

$$\text{The Slope: } m_{\tan} = f'(a).$$

The Equation of Tangent Line: Now we have all the "input data" we need to utilize (S-2) Taking

$$(\, x_0, y_0 \,) = (\, a, f(a) \,) \qquad m = m_{\,\mathrm{tan}} = f'(a),$$

and inserting this information into (S-2) we obtain

$$y - f(a) = f'(a)(x - a).$$

or

$$\boxed{y = f(a) + f'(a)(x - a).}$$

This verifies (5).

Example 3.2. ■

3.3. We just "plug" our information into the general formula developed in (5). That general formula is

$$y = f(a) + f'(a)(x - a).$$

This is the equation of the line tangent to the graph of f at $x = a$.

Our function is $f(x) = 3x^2$, our point of interest is $x = 1$ (this is our a, i.e. $a = 1$).

Auxiliary Calculations:

$$\begin{aligned} a &= 1 \\ f(1) &= 3(1)^2 = 3 \\ f'(1) &= 6 \qquad \text{from Example 3.1} \end{aligned}$$

The Equation of Tangent Line: Now we have all the "input data" we need to utilize (5): Taking and inserting this information into (5) we obtain

$$y = 3 + 6(x - 1).$$

or

$$\boxed{y = 6x - 3.}$$

This is the equation of the line tangent to the graph of f at $x = 1$.

Example 3.3. ■

3.4. We must build up the expression needed to calculate the derivative from the definition. We refer you to (2). Let h be a symbol that represents a nonzero number, then

$$\begin{aligned} f(a) &= f(2) = 4 \\ f(a+h) &= f(2+h) = (2+h)^2 \\ &= 4 + 4h + h^2 \\ f(a+h) - f(a) &= 4 + 4h + h^2 - 4 \\ &= 4h + h^2. \end{aligned}$$

Thus,

$$\frac{f(a+h) - f(a)}{h} = \frac{4h + h^2}{h} = 4 + h,$$

where we have cancelled the common factor of h.

Finally,

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} 4 + h \\ &= 4. \end{aligned}$$

Thus,

$$\boxed{f'(2) = 4.}$$

That was simple!

Example Notes: Study the style of presentation, the organization of this solution. If you are well-organized, methodical, and develop good habits, then the calculation of the derivative from the definition is straight forward. Is it something you would like to strive for?

▪ Derivatives at *particular* values of x, such as $x = 2$, are fine. But now, if we wanted the derivative at $x = 3$ we are out of luck. All our work is for nothing! We have recalculate everything for the case

$x = 3$; to avoid the unpleasant situation, it is best to obtain a general formula for the derivative at *any* value of x.

▪ The interpretation of $f'(2) = 4$ is that it is the slope of the line tangent to the graph of f at $x = 2$ ▪

Example 3.4. ▪

3.5. This is a continuation of EXAMPLE 3.4. From that example, $f'(2) = 4$. This represents the slope the of the line tangent to the graph of f at the point $x = 2$.

Now the find the equation of a line, we need the slope of that line ($m = f'(2) = 4$), and we need a point on that line. The point on the line is the point of tangency:

$$P(\,a, f(a)\,) = P(\,2, f(2)\,) = P(\,2, 4\,).$$

We now have all the information we need. The point-slope form of the equation of a line is

$$y - y_0 = m(x - x_0),$$

where, the line has slope m and passes through the point $(\,x_0, y_0\,)$. Substituting in our data,

$$y - 4 = 4(x - 2)$$

or,

$$\boxed{y = 4x - 4}$$

is the desired equation. Example 3.5. ■

3.6. Begin with the difference quotient *and* simplify. Here, $f(x) = \sqrt{2x}$ and $a = 3$ in the definition.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{f(3+h) - f(3)}{h} \\ &= \frac{\sqrt{2(3+h)} - \sqrt{6}}{h} \qquad \text{Are you following?} \\ &= \frac{\sqrt{6+2h} - \sqrt{6}}{h} \\ &= \frac{\sqrt{6+2h} - \sqrt{6}}{h} \, \frac{\sqrt{6+2h} + \sqrt{6}}{\sqrt{6+2h} + \sqrt{6}} \end{aligned}$$

In the last step we utilize the conjugate trick (remember?). The reason I went for the conjugate trick here is because of the basic *Differentiation Strategy*. The thing what was guiding my thinking was the goal of getting rid of the h in the denominator (legally!). The conjugate trick did the trick!

Continuing now ...

$$\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{6+2h}-\sqrt{6}}{h}\,\frac{\sqrt{6+2h}+\sqrt{6}}{\sqrt{6+2h}+\sqrt{6}}$$
$$= \frac{(6+2h-6}{h(\sqrt{6+2h}+\sqrt{6})}$$
$$= \frac{2h}{h(\sqrt{6+2h}+\sqrt{6})}$$
$$= \frac{2}{\sqrt{6+2h}+\sqrt{6}} \quad \text{cancel the } h\text{'s!}$$

Success vis–á–vis *Differentiation Strategy*! And finally,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{6+2h)}+\sqrt{6}} \\ &= \frac{2}{\sqrt{6}+\sqrt{6}} \\ &= \frac{2}{2\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}. \end{aligned}$$

How's that for micro-miniature steps!

If $f(x)=\sqrt{2x}$, then $\boxed{f'(x)=\frac{1}{\sqrt{6}}.}$

Example 3.6. ■

3.7. This really isn't Skill Level 5, it's easier than that. You just have to have the courage to do the obvious. Let's calculate the infamous *difference quotient*:

$$\frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h} = \begin{cases} \dfrac{h^2}{h} & \text{if } h \text{ is rational} \\ 0 & \text{if } h \text{ is irrational} \end{cases}$$

since $f(0) = 0$ (0 is a rational number). Thus the difference quotient is then

$$\frac{f(0+h)-f(0)}{h} = \begin{cases} h & \text{if } h \text{ is rational} \\ 0 & \text{if } h \text{ is irrational} \end{cases} \tag{S-3}$$

Now, based on the form of the difference quotient, and the fact that we want to take the limit as h goes to 0, we would postulate that $f'(0) = 0$. But we must prove this still. We want to prove

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0.$$

This is a job for the $\delta\epsilon$-definition of limit. Let $\epsilon > 0$, and choose $\delta = \epsilon$. (Why? Because it works, silly — bottom line.)

Now suppose h is such that

$$0 < |h - 0| = |h| < \delta.$$

Either that h is a rational number or a irrational number.

Case 1: Suppose h is rational, and $0 < |h| < \delta$. Then

$$\left| \frac{f(0+h) - f(0)}{h} \right| = |h| < \delta = \epsilon. \tag{S-4}$$

The first equality comes from (S-3).

Case 2: Suppose h is irrational, and $0 < |h| < \delta$. Then by (S-3),

$$\left| \frac{f(0+h) - f(0)}{h} \right| = 0 < \epsilon. \tag{S-5}$$

Conclusion: From (S-4) and (S-5), either case implies

$$\left| \frac{f(0+h) - f(0)}{h} \right| < \epsilon$$

whenever, $0 < |h| < \delta$. But this is exactly the definition of what it means for

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0$$

But by definition of derivative,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Thus, we have proved

$$\boxed{f'(0) = 0.}$$

Example Notes: In fact, $x = 0$ is the *only* point at which this function f has a derivative. That would be interesting to prove, wouldn't it? Take that as a challenge. 𝔇𝔓 ■

Example 3.7. ■

3.8. We must build up the difference quotient, keeping x symbolic.

Initial Calculations:

$$\begin{aligned} f(x) &= x^2 \\ f(x+h) &= (x+h)^2 \\ &= x^2 + 2hx + h^2 \\ f(x+h) - f(x) &= (x^2 + 2hx + h^2) - x^2 \\ &= 2hx + h^2 \end{aligned}$$

Difference Quotient:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2hx + h^2}{h} \\ &= 2x + h. \end{aligned}$$

Notice, I have successfully carried out our *Differentiation Strategy*. Finally, *The Derivative*:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} 2x + h \\ &= 2x. \end{aligned}$$

$$\boxed{f'(x) = 2x.}$$

Example Notes: As you can see, $f'(x) = 2x$ is a function whose name is f'. In Example 3.4, we calculated $f'(2) = 4$ for the exact same function. This taught us something at the time, but if we wanted to know $f'(3)$ we would have to recalculate everything. The charm of the derivative function is that we don't have to recalculate each time. We showed that $f'(x) = 2x$ so $f'(2) = 2(2) = 4$, and $f'(3) = 2(3) = 6$. No pain!

- Note also that $\mathrm{Dom}(f') = \mathrm{Dom}(f) = \mathbb{R}$. ■

Example 3.8. ■

3.9. Begin with the difference quotient *and* simplify. Here, $f(x) = \sqrt{2x}$ and $a = 2$ in the definition.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\
&= \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \\
&= \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \, \frac{\sqrt{2x+2h} + \sqrt{2x}}{\sqrt{2x+2h} + \sqrt{2x}}
\end{aligned}$$

In the last step we utilize the conjugate trick (remember?). Continuing now . . .

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{2x+2h}-\sqrt{2x}}{h}\,\frac{\sqrt{2x+2h}+\sqrt{2x}}{\sqrt{2x+2h}+\sqrt{2x}} \\
&= \frac{(2x+2h)-2x}{h(\sqrt{2x+2h}+\sqrt{2x})} \\
&= \frac{2h}{h(\sqrt{2x+2h}+\sqrt{2x})} \\
&= \frac{2}{\sqrt{2x+2h}+\sqrt{2x}} \quad \text{cancel the } h\text{'s!}
\end{aligned}
$$

Success vis–á–vis *Differentiation Strategy*! And finally,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h)}+\sqrt{2x}} \\ &= \frac{2}{\sqrt{2x}+\sqrt{2x}} \\ &= \frac{2}{2\sqrt{2x}} \\ &= \frac{1}{\sqrt{2x}}. \end{aligned}$$

Thus, the derivative of the function $f(x)=\sqrt{2x}$ is

$$\boxed{f'(x)=\frac{1}{\sqrt{2x}}.} \tag{S-6}$$

Particular values of the derivative can be obtained by simple evaluation,

$$f'(3) = \frac{1}{\sqrt{6}} \qquad f'(2) = \frac{1}{2} \qquad f'(3/2) = \frac{2}{\sqrt{6}}$$

Domain Analysis: The domain of the parent function is $\text{Dom}(f) = [\,0, +\infty\,)$, but $\text{Dom}(f') = (\,0, +\infty\,)$. Compare the two domains, the only difference is the exclusion of $x = 0$ from the $\text{Dom}(f')$. There are two reasons for this: (1) On technical grounds, the Definition 3.1 requires that if we are to differentiate at $x = a$, then a is required to lie in an open interval of the domain of f. This is necessary to calculate a two-sided limit. But $x = 0$ is an endpoint of the domain, and is therefore excluded on technical grounds. One-sided derivatives address this problem. And (2), when you put $x = 0$ into the derivative formula (S-6), you get 0 in the denominator.

Example 3.9. ■

3.10. We will use Corollary 3.3. Argue that f is not continuous at $x = 0$.

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x+1 & x > 0 \end{cases}$$

Left-hand limit:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0 = f(0) \tag{S-7}$$

Right-hand limit:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x + 1 = 1 \neq f(0) \tag{S-8}$$

Therefore, $\lim_{x \to 0} f(x)$ does not exist; consequently, f cannot be continuous. (Other references (S-7) states that f is left-continuous at $x = 0$, (S-8) implies that f is *not* right-continuous at $x = 0$; therefore f is not continuous at $x = 0$.)

Example 3.10. ■

3.11. We show that the left-hand derivative $f'_{-}(0) = -1$ and the right-hand derivative $f'_{+}(0) = 1$. It follows then, by Theorem 3.5, that $f'(0)$ does not exist.

First note that if $h < 0$, $f(h) = |h| = -h$; and if $h > 0$, $f(h) = |h| = h$. Of course $f(0) = |0| = 0$.

Left-hand derivative:

$$\begin{aligned} f'_{-}(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} \frac{-h}{h} \\ &= \lim_{h \to 0^-} -1 \\ &= -1. \end{aligned}$$

As advertised, $\boxed{f'_{-}(0) = -1}$.

Right-hand derivative:

$$\begin{aligned} f'_+(0) &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{h}{h} \\ &= \lim_{h\to 0^+} 1 \\ &= 1. \end{aligned}$$

Again, $\boxed{f'_+(0) = 1}$.

We have shown that

$$f'_-(0) = -1 \neq 1 = f'_+(0),$$

and conclude, by Theorem 3.5, that $f'(0)$ *does not exist.*

Example 3.11. ■

3.12. We reason as follows.

Left-hand derivative:

$$\begin{aligned} f'_-(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0^-} 0 \\ &= 0 \end{aligned}$$

Thus, $\boxed{f'_-(0) = 0}$.

Right-hand derivative: We can compute the right-hand derivative from its definition, but let me illustrate a different reasoning.

Consider the function $g(x) = x^2$, $x \in \mathbb{R}$. Now, we have shown in Example 3.8 that $g'(x) = 2x$. Also, $f(x) = g(x)$ for all $x \geq 0$. The right-hand derivative at $x = 0$ is determined by the value of the function to the right of $x = 0$. Therefore, we conclude $f'_+(0) = g'_+(0)$!

(Think about it.) Apply Theorem 3.5 to the function g to obtain $g'(0) = g'_-(0) = g'_+(0)$. We deduce

$$f'_+(0) = g'_+(0) = g'(0) = 2(0) = 0.$$

Thus, $\boxed{f'_+(0) = 0}$.

We have shown that $f'_-(0) = 0 = f'_+(0)$ and so conclude that $f'(0)$ exists and

$$f'(0) = 0.$$

Example 3.12. ■

3.13. It is easy to show that $f'_-(0) = -\infty$ and $f'_+(0) = +\infty$.

Difference Quotient: We are interested in the case of $x = 0$ in the formula for the difference quotient:

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{f(h) + f(0)}{h} \\ &= \frac{h^{2/3}}{h} = h^{-1/3} \\ &= \frac{1}{h^{1/3}} \qquad \text{(S-9)}\end{aligned}$$

Left-hand Derivative: The left-hand derivative is the left-hand limit of the difference quotient, from (S-9)

$$f'_-(0) = \lim_{h \to 0^-} \frac{1}{h^{1/3}} = -\infty.$$

This calculation is enough to show that f is not differentiable at $x = 0$ because, for differentiable functions, the left-hand derivative is required to be finite.

Right-hand Derivative: The right-hand derivative is the right-hand limit of the difference quotient, from (S-9),

$$f'_+(0) = \lim_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty.$$

Example 3.13. ■

6.1. In light of the Power Rule, this is the height of triviality!

$$\frac{d\,x^4}{dx} = 4x^3 \qquad \frac{d\,x^{123}}{dx} = 123x^{122}$$
$$\frac{d\,s^5}{ds} = 5x^4 \qquad \frac{d\,w^{10}}{dw} = 10w^9$$

When you do problems, use correct notation. Don't be lazy. Pretend that I am looking over your shoulder watching your work. Be neat. Be organized. Example 6.1. ■

6.2. There is no big secret to this "proof." We just use standard techniques.

$$\begin{aligned} f(x) &= x^4 \\ f(x+h) &= (x+h)^4 \\ f(x+h) - f(x) &= (x+h)^4 - x^4 \\ &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4 \\ &= 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \qquad \text{(S-10)} \end{aligned}$$

The difference quotient is

$$\frac{f(x+h) - f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3,$$

where I have already cancelled out the h in the denominator with the common factor of h in the numerator (S-10) — forgive me for doing it behind your back.

Finally,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \\ &= 4x^3. \end{aligned}$$

Thus we have shown that for $f(x) = x^4$, $f'(x) = 4x^3$. You'll note that this formula is consistent with the general formula of the power rule.

Example 6.2. ■

7.1. A simple application of the Power Rule

$$\frac{d\,4x^6}{dx} = 4\frac{d\,x^6}{dx} = 4(6x^5) = 24x^5$$

$$\frac{d\,21s^3}{ds} = 21\frac{d\,s^3}{ds} = 21(3s^2) = 63s^2$$

$$\frac{d\,\frac{5}{4}w^4}{dw} = \frac{5}{4}\frac{d\,w^4}{dw} = \frac{5}{4}(4w^3) = 5w^3,$$

where we have used the Homogeneous Property and the Power Rule in each calculation.

Example Notes: Notice how the Leibniz notation is used to work through this problem. In the above links, I have referenced the *verbal* versions of the rules. Each time you use a formula, *verbalize*! Your audio output is fed back into your audio sensors — the result: You hear yourself speaking the formula. Through this device, you *will* automatically *memorize* these important formulas.

■

Example 7.1. ■

7.2. Proceed as follows:

$$\begin{aligned}
\frac{d}{dx}(3x^4 - 6x^5) &= \frac{d}{dx}(3x^4) - \frac{d}{dx}(6x^5) && \triangleleft \text{ Add. Prop.} \\
&= 3\frac{d\,x^4}{dx} - 6\frac{d\,x^5}{dx} && \triangleleft \text{ Homogen. Prop.} \\
&= 3(4x^3) - 6(5x^4) && \triangleleft \text{ Power Rule} \\
&= 12x^3 - 30x^4.
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}(3x^4 - 6x^5) = 12x^3 - 30x^4.}$$

Example Notes: Notice how the Leibniz notation is used to work through this more "complicated" problem.

▪ As you are using the different differentiation properties, *verbalize them*. The sound goes out the mouth, around your head, and into your ears. As you do many differentiation problems, verbalizing as you go, you will soon memorize these important formulas without effort. ■

Example 7.2. ■

7.3. We use standard techniques ... do you?

$$\begin{aligned}\frac{d}{dw}3\csc(w) - 5\cot(w) &= \frac{d}{dw}3\csc(w) - \frac{d}{dw}5\cot(w)\\ &= 3\frac{d}{dw}\csc(w) - 5\frac{d}{dw}\cot(w)\\ &= 3(-\csc(w)\cot(w)) - 5(-\csc^2(w))\\ &= -\csc(w)(3\cot(w) + 5\csc(w))\end{aligned}$$

Example Notes: Use the Leibniz notation to help you work through a problem. Be slow and methodical. *Verbalize* the rules as you use them — this will help you remember them. ■

Example 7.3. ■

7.4. This is the derivative of a product, so we apply the Product Rule. I want to be true to my own self — I'll use good algebraic and calculus methods,

$$\begin{aligned}
\frac{d}{dx} x^3(2x^9 - 12) &= x^3 \frac{d}{dx}(2x^9 - 12) + (2x^9 - 12)\frac{d\,x^3}{dx} && \triangleleft \text{ Prod. Rule} \\
&= x^3(18x^8) + (2x^9 - 12)(3x^2) && \triangleleft \text{ Power Rule} \\
&= 18x^{11} + 3x^2(2x^9 - 12) \\
&= 3x^2(6x^9 + 2x^9 - 12) \\
&= 3x^2(8x^9 - 12) \\
&= 12x^2(2x^9 - 3).
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx} x^3(2x^9 - 12) = 12x^2(2x^9 - 3).}$$

Example Notes: Use every opportunity to practice your algebra. Simply multiplying out an algebraic expression does not (necessarily) constitute a simplification. Study the presentation style above. Use the Leibniz notation to help you work through a long problem. *Verbalize* the formulas as you use them. ■

Example 7.4. ■

7.5. We are asked to differentiate a quotient; therefore, we apply the Quotient Rule.

$$\begin{aligned} \frac{d}{dx}\frac{3x^3}{4-5x^2} &= 3\frac{d}{dx}\frac{x^3}{4-5x^2} && \triangleleft \text{ Homogen.} \\ &= 3\frac{(4-5x^2)\dfrac{d}{dx}(x^3) - x^3\dfrac{d}{dx}(4-5x^2)}{(4-5x^2)^2} && \triangleleft \text{ Quot. Rule} \\ &= 3\frac{(4-5x^2)(3x^2) - x^3(-10x)}{(4-5x^2)^2} && \triangleleft \text{ Add., Power} \\ &= \frac{3x^2(3(4-5x^2)+10x^2)}{(4-5x^2)^2} \end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}\frac{3x^3}{4-5x^2} = \frac{3x^2(12-5x^2)}{(4-5x^2)^2}.}$$

Example Notes: Again, note the use of the notation to work through this problem. Use good notation to help you! Verbalize the formulas as you use them. The hyper-links provided above point to the formulas in their verbalized form. Read them. ■

Example 7.5. ■

7.6. The variable w is the dependent variable and s is the independent variable. Apply the Quotient Rule, and *verbalize* as we go.

$$\begin{aligned}\frac{dw}{ds} &= \frac{d}{ds}\frac{s^2}{3s^4-2} \\ &= \frac{(3s^4-2)\dfrac{d}{ds}s^2 - s^2\dfrac{d}{ds}(3s^4-2)}{(3s^4-2)^2} \\ &= \frac{(3s^4-2)(2s) - s^2(12s^3)}{(3s^4-2)^2} \\ &= \frac{2s(3s^4-2-6s^4)}{(3s^4-2)^2} \\ &= -\frac{2s(3s^4+2)}{(3s^4-2)^2}.\end{aligned}$$

Throughout, I have used the Power Rule, the Quotient Rule, sprinkled with a liberal dose of standard *algebra*. Study the calculus and the algebra.

$$\boxed{\frac{dw}{ds} = -\frac{2s(3s^4+2)}{(3s^4-2)^2}.}$$

Example 7.6. ■

7.7. Students make a big deal out of this kind of problem. Very often they use the quotient rule.

$$\begin{aligned}
\frac{d}{dx}\frac{5}{x^4} &= \frac{x^4\,\dfrac{d\,5}{dx} - 5\,\dfrac{d\,x^4}{dx}}{(x^4)^2} \\
&= \frac{x^4(0) - 5(4x^3)}{x^8} \\
&= -\frac{20x^3}{x^8} \\
&= -\frac{20}{x^5}
\end{aligned}$$

This yields the right answer *but it is not the best and most efficient way to solve this problem. Don't do it this way*!

The optimal solution is

$$\begin{aligned}
\frac{d}{dx}\frac{5}{x^4} &= 5\frac{d\,x^{-4}}{dx} && \triangleleft \text{ Homogen.} \\
&= 5(-4)x^{-5} && \triangleleft \text{ Power Rule} \\
&= -\frac{20}{x^5}
\end{aligned}$$

This solution is quicker and there is a lesser chance of an error. The quotient rule is a complicated rule. The more complicated the rule, the greater chance of error.

Example Lesson: When you have a quotient, and either the numerator or denominator is a constant, then don't use the quotient rule. Use the Power Rule.

Example Notes: This same kind of mistake occurs with a product. Some students will apply the product rule to the function $y = 5x^3$. Technically, there is nothing wrong with doing it that way but doing so would be wasting a lot of time on a problem that could be done by inspection $y' = 15x^2$. ∎

Example 7.7. ■

7.8. It is very tempting to use the product rule to evaluate $\dfrac{d}{dx}x\sqrt{x}$; however, it would be inefficient to do so. The key point here is to realize that

$$x\sqrt{x} = xx^{1/2} = x^{3/2}.$$

Then

$$\frac{d}{dx}x\sqrt{x} = \frac{d\,x^{3/2}}{dx} = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}.$$

Example Lesson: When approaching a differentiation problem, look first at the function to be differentiated. Determine whether there is a worthwhile simplification *before* differentiating.

Example 7.8. ■

7.9. Now we will have to use the Quotient Rule.

$$\begin{aligned}
\frac{d}{dx}\frac{x^2+1}{\sqrt{x}+1} &= \frac{d}{dx}\frac{x^2+1}{x^{1/2}+1} \\
&= \frac{(x^{1/2}+1)(2x)-(x^2+1)(1/2)x^{-3/2}}{(x^{1/2}+1)^2} \\
&= \frac{(1/2)x^{-3/2}(x^{1/2}+1)(4x^{5/2})-(x^2+1)}{(x^{1/2}+1)^2} \\
&= \frac{x^{-3/2}(4x^3+4x^{5/2}-x^2-1)}{2(x^{1/2}+1)^2} \\
&= \frac{4x^3+4x^{5/2}-x^2-1}{2x^{3/2}(x^{1/2}+1)^2} \\
&= \frac{4x^3+4x^2\sqrt{x}-x^2-1}{2x\sqrt{x}(\sqrt{x}+1)^2}
\end{aligned}$$

Example 7.9. ■

7.10. The problem is to calculate $d(x^2 \sin(x))$. This is the differential of a *product*, we'll use the product rule:

$$\begin{aligned} d(x^2 \sin(x)) &= x^2\, d(\sin(x)) + \sin(x)\, dx^2 \\ &= x^2 (\cos(x)\, dx) + \sin(x)(2x\, dx) \\ &= x^2 \cos(x)\, dx + 2x \sin(x)\, dx \\ &= (x^2 \cos(x) + 2x \sin(x))\, dx \end{aligned}$$

Thus,

$$\boxed{d(x^2 \sin(x)) = (x^2 \cos(x) + 2x \sin(x))\, dx.}$$

If we divide both sides by dx, we get

$$\frac{d\, x^2 \sin(x)}{dx} = x^2 \cos(x) + 2x \sin(x),$$

which is the correct derivative.

Example 7.10. ■

7.11. The equation $dy = 6\,dx$ can be interpreted as the equation of the tangent line in the following way.

Derivative: $y = x^2$, $\dfrac{dy}{dx} = 2x$.

Differential: $dy = 2x\,dx$

Point of Tangency:

$$y = x^2 \text{ and } x = 3 \implies y = 3,$$

Thus, the point of tangency is $(3, 9)$.

Slope of Tangent Line:

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{x=3} = 2x\big|_{x=3} = 6.$$

Differential at $x = 3$: $dy = 6\,dx$.

Now, translate the xy-axis system to the point of tangency: $(3, 9)$. Call the new, translated axis system the $dx\,dy$-axis system; i.e. call the new horizontal axis, dx, and the new vertical axis dy. The equation of any line through the new origin is

$$dy = m\,dx,$$

where m is the slope of the a line through the new origin.

The tangent line under consideration passes through the point of tangency, $(\,3, 9\,)$, which, by design, is our new origin. The slope of this line is $m = 6$, thus, the equation of the tangent line, in our new axis system, is given by

$$dy = 6\,dx. \tag{S-11}$$

This represents another interpretation of the differential: It is the equation of the tangent line, when the xy-axis system is translated to the point of tangency, and the new axis variables are dx and dy.

If you remember your *translation of axis knowledge*, then you can write down the *transformation equations*: In general, if we move the origin from $(0, 0)$ to (x_0, y_0), and we are calling the new axis dx and dy, then

$$dx = x - x_0$$
$$dy = y - y_0$$

In our case, we have moved the origin to $(\, x_0, y_0\,) = (\, 3, 9\,)$. Thus, the transformation of axes equations becomes,

$$dx = x - 3$$
$$dy = y - 9$$

Now if we substitute these equation back into (S-11), we get

$$y - 9 = 6(x - 3)$$

or,

$$y = 9 + 6(x - 3)$$

These equations are that of the tangent line, back in the original xy-coordinate system. This can be verifies by "plugging" in the basic information into the general formula.

Summary of Technique. Let $y = f'(x)$. Put $x = x_0$ a numerical value. Then $dy = f'(x_0)\,dx$ is the equation of the line tangent to the graph of f, then the xy-axis system is translated to the point of tangency: $(\,x_0, y_0\,)$, $y_0 = f(x_0)$. The new axis system has axis variables, dx and dy. The equation of the tangent line can be obtained by taking

$$dy = f'(x_0)\,dx,$$

and translating back to the original xy-axis. To translate back, use the transformation equations,

$$dx = x - x_0 \qquad dy = y - y_0.$$

Thus,

$$(y - y_0) = f'(x_0)(x - x_0).$$

Example 7.11. ■

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