

Global Instructions: Solve each of the following problems without error. Box in your answers. Be neat and organized.

1. Evaluate the integral $\int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz$.

Solution: The calculation goes as follows:

$$\begin{aligned} \int_0^1 \int_0^z \int_0^{x+z} 6xz \, dy \, dx \, dz &= \int_0^1 \int_0^z 6xz y \Big|_0^{x+z} \, dx \, dz = \int_0^1 \int_0^z 6xz(x+z) \, dx \, dz \\ &= \int_0^1 \int_0^z 6z(x^2 + xz) \, dx \, dz = \int_0^1 6z \left(\frac{1}{3}x^3 + \frac{1}{2}x^2z \right) \Big|_0^z \, dz \\ &= \int_0^1 6z \left(\frac{1}{3}z^3 + \frac{1}{2}z^3 \right) \, dz = \int_0^1 2z^4 + 3z^4 \, dz \\ &= \int_0^1 5z^4 \, dz = z^5 \Big|_0^1 = \boxed{1} \end{aligned}$$

2. Consider the integral: $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx$. Sketch the region of integration and label the boundary surfaces.

Solution: Left to the student.

3. *Set up* the triple integral for the volume of the solid in the first octant bounded by the cylinder $y^2 + z^2 = 1$ and by the planes $y = x$, $x = 0$ and $z = 0$.

Solution: For example,

$$V = \int_0^1 \int_0^y \int_0^{\sqrt{1-z^2}} dz \, dx \, dy$$

4. *Set up* the limits of integration for $\iiint_E f(x, y, z) \, dV$, where E is the region formed by the planes $\frac{x}{4} + \frac{y}{2} + \frac{z}{3} = 1$, $x = 0$, $y = 0$, and $z = 0$.

- (a) Take $dV = dz \, dx \, dy$.

Solution: We have,

$$\iiint_E f(x, y, z) \, dV = \int_0^2 \int_0^{4(1-y/2)} \int_0^{3(1-x/4-y/2)} dz \, dx \, dy$$

- (b) Take $dV = dy \, dx \, dz$.

Solution:

$$\iiint_E f(x, y, z) \, dV = \int_0^3 \int_0^{4(1-z/3)} \int_0^{2(1-x/4-z/3)} dy \, dx \, dz$$

5. Calculate the Jacobian of the transformation $\begin{cases} x = u^3 - 3v \\ y = 2u + v^3 \end{cases}$.

Solution: See simple enough, but is it a trick question?

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{vmatrix} 3u^2 & -3 \\ 2 & 3v^2 \end{vmatrix} \right| = \boxed{9u^2v^2 + 6}$$

6. Convert the following integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2} dz dy dx$ to cylindrical coordinates. Do not evaluate.

Solution: First, visualize/draw the region of integration, then see that

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2} dz dy dx = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} r^4 dz dr d\theta$$

7. Set up the triple integral in spherical coordinates for the volume of the region E in the first octant bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$.

Solution:

$$V(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) d\rho d\theta d\phi$$

8. Do a change of variables on the integral $\iint_{\mathcal{R}} dA$, using $\begin{cases} x = u \\ y = 3v \end{cases}$, where \mathcal{R} be the region bounded by $9x^2 + y^2 = 36$.

(a) Calculate the Jacobian of this transformation.

Solution: The Jacobian is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \right| = \boxed{3}$$

(b) Find the equation that represents the boundary of the new region, S , in the uv -plane.

Solution: Follow the boundary,

$$9x^2 + y^2 = 36 \implies 9u^2 + (3v)^2 = 36 \implies 9u^2 + 9v^2 = 36 \implies \boxed{u^2 + v^2 = 4}$$

(c) Using parts (a) and (b), evaluate $\iint_{\mathcal{R}} dA$. (Note: Change to polar coordinate variables in the uv -plane.)

Solution:

$$\iint_{\mathcal{R}} dA = \int_0^{2\pi} \int_0^2 r(3) dr d\theta = 6\pi \int_0^2 r dr = 6\pi \cdot \frac{1}{2} r^2 \Big|_0^2 = 6\pi \cdot \frac{1}{2}(4) = \boxed{12\pi}$$

9. Calculate the integral $\int_{\mathcal{C}} y \, ds$, where \mathcal{C} is that portion of the circle $x^2 + y^2 = 4$ in the first quadrant.

Solution: The standard parameterizations for the circle is

$$\begin{cases} x = 2 \cos(t) \\ y = 2 \sin(t) \end{cases}, \quad 0 \leq t \leq \pi/2$$

It is easy to calculate $ds = 2 \, dt$. The evaluation of the line integral is then

$$\int_{\mathcal{C}} y \, ds = \int_0^{\pi/2} 2 \sin(t) 2 \, dt = -4 \cos(t) \Big|_0^{\pi/2} = \boxed{4}$$

10. Evaluate the integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = x^2y\mathbf{i} - xy\mathbf{j}$, and \mathcal{C} is given by $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $0 \leq t \leq 1$.

Solution: We have

$$\begin{aligned} d\mathbf{r} &= (3t^2\mathbf{i} + 4t^3\mathbf{j}) \, dt \\ \mathbf{F}(\mathbf{r}(t)) &= (t^3)^2(t^4)\mathbf{i} - (t^3)(t^4)\mathbf{j} = t^{10}\mathbf{i} - t^7\mathbf{j} \\ \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= 3t^{12} - 4t^{10} \, dt \end{aligned}$$

Finally,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_0^1 3t^{12} - 4t^{10} \, dt = \frac{3}{13} - \frac{4}{11}$$

11. Calculate the value of the line integral $\int_{\mathcal{C}} x^2y \, dy$, where \mathcal{C} is the line segment from $(1, 2, 3)$ to $(-2, 3, -1)$.

Solution: The initial problem is to find parametric equations representing the path. We use our knowledge from Chapter 13. Let $\mathbf{v} = \langle -2, 3, -1 \rangle - \langle 1, 2, 3 \rangle = \langle -3, 1, -4 \rangle$, thus

$$\mathcal{C} : \begin{cases} x = 1 - 3t \\ y = 2 + t \\ z = 3 - 4t \end{cases} \quad 0 \leq t \leq 1, \quad \text{and} \quad \begin{cases} dx = -3 \, dt \\ dy = dt \\ dz = -4 \, dt \end{cases}$$

The line integral calculation follows:

$$\begin{aligned} \int_{\mathcal{C}} x^2y \, dy &= \int_0^1 (1 - 3t)^2(2 + t) \, dt \\ &= \int_0^1 2 - 11t + 12t^2 + 9t^3 \, dt \\ &= 2 - \frac{11}{2} + 4 + \frac{9}{4} = \boxed{\frac{11}{4}} \end{aligned}$$

12. Let $f(x, y) = \frac{1}{2}x^2y^4$.

(a) Let $\mathbf{F} = \nabla f$, find \mathbf{F} .

$$\text{Solution: } \mathbf{F} = \nabla f = \boxed{xy^4\mathbf{i} + 2x^2y^3\mathbf{j}}$$

(b) Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where \mathcal{C} is given by $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $0 \leq t \leq 1$.

Solution: \mathbf{F} is a conservative vector field with a potential function $f(x, y) = \frac{1}{2}x^2y^4$. The path goes from the point $(0, 0)$ to $(1, 1)$. The integral is path independent, hence,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(0, 0) = \boxed{\frac{1}{2}}$$

That was easy enough.

13. Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field.

(a) State a condition equivalent to \mathbf{F} being a *conservative* vector field.

Solution:

$$\frac{\partial}{\partial y}P = \frac{\partial}{\partial x}Q$$

(b) Determine whether or not the function $\mathbf{F}(x, y) = (x^2 + 2xy)\mathbf{i} + (y + x^2)\mathbf{j}$ is a conservative vector field, and if conservative, find a potential function f .

Solution: The vector function \mathbf{F} is a conservative since

$$(\partial/\partial y)(x^2 + 2xy) = 2x = (\partial/\partial x)(y + x^2)$$

Let f be potential function of \mathbf{F} . Then $f_x = x^2 + 2xy$ and $f_y = y + x^2$. then, holding y fixed, we have,

$$f(x, y) = \int f_x = \int x^2 + 2xy dx = \frac{1}{3}x^3 + x^2y + g(y) \quad (1)$$

Now

$$y + x^2 = f_y = \frac{\partial}{\partial y}x^3 + x^2y + g(y) = x^2 + g'(y)$$

Thus, $g'(y) = y$ and so $g(y) = \frac{1}{2}y^2$. Putting this back into equation (1), we get

$$\boxed{f(x, y) = \frac{1}{3}x^3 + x^2y + \frac{1}{2}y^2}$$